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Holographic nonlinear hydrodynamics from AdS/CFT with multiple/non-Abelian symmetries

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Abstract

We study viscous hydrodynamics of hot conformal field theory plasma with multiple/non-Abelian symmetries in the framework of AdS/CFT correspondence, using a recently proposed method of directly solving bulk gravity in derivative expansion of local plasma parameters. Our motivation is to better describe the real QCD plasma produced at RHIC, incorporating its $U(1)^{N_f}$ flavor symmetry as well as $SU(2)_I$ non-Abelian iso-spin symmetry. As concrete examples, we choose to study the STU model for multiple $U(1)^3$ symmetries, which is a sub-sector of 5D N=4 gauged SUGRA dual to N=4 Super Yang-Mills theory, capturing Cartan $U(1)^3$ dynamics inside the full R-symmetry. For $SU(2)$, we analyze the minimal 4D N=3 gauged SUGRA whose bosonic action is simply an Einstein-Yang-Mills system, which corresponds to $SU(2)$ R-symmetry dynamics on M2-branes at a Hyper-Kahler cone. By generalizing the bosonic action to arbitrary dimensions and Lie groups, we present our analysis and results for any non-Abelian plasma in arbitrary dimensions.

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1 Introduction and summary

Strongly coupled plasma of finite temperature gauge theories has recently become a fascinating subject of research, largely motivated by the RHIC experiment of relativistic heavy ion collisions. Naive QCD expectations based on perturbative QCD have failed to explain certain important aspects of the created QCD plasma, and there are several indications that the RHIC plasma is in fact a strongly coupled liquid. Given the situation, one may hope that the problem can be attacked by AdS/CFT correspondence or gauge/gravity correspondence because the correspondence is useful precisely when the gauge theory side is strongly coupled [1]. In the gravity side, a finite temperature plasma corresponds to a black-hole, or more precisely black-brane, spacetime with Hawking temperature identified with the temperature of the gauge theory. The black-hole horizon is located at certain point in the holographic additional dimension, and presumably physics outside the horizon with suitable boundary conditions on the horizon describes the finite temperature plasma of gauge theories. There have been a lot of useful and often surprising results obtained from this gravity picture, which would be hard to be found in the pure gauge theory analysis due to strong coupling [2, 3, 4, 8, 9, 10].

Of course, the main huddle that hinders further progress in this direction is the absence of precise dual gravity theory of the real QCD at present, although it is important to improve the current models to better mimic realistic QCD [11, 12, 13, 14]. However, certain properties of strongly coupled finite temperature plasma may be universal at least qualitatively [4]; a well-known example is the viscosity-entropy ratio [2, 3, 4], $\eta/s \sim \frac{1}{4\pi}$, which in fact is close to the RHIC experiment data, and it is not a vividly wrong idea to try to learn something about realistic RHIC plasma by studying certain specific AdS/CFT models at finite temperature[1]. The question whether the results obtained in the specific model are meaningful in realistic QCD should be asked carefully though.

Any finite temperature plasma is described by hydrodynamics in sufficiently slowly-varying and long-ranged regime. It is more a framework rather than a result; it is based on local thermal equilibrium and conservation of symmetry, such as energy-momentum or global symmetry currents. It is then natural to expect that hydrodynamics should be emerging in the gravity side of finite temperature plasma described by black-branes. Indeed, it is based on this idea that various hydrodynamic coefficients including the viscosity-entropy ratio were calculated via linear response theory in the gravity back-

[1] See [5] for possible violations of viscosity-entropy bound.

ground [2, 3, 4, 6, 7, 10, 15]. *Ab initio* way of deriving the hydrodynamics from the gravity side when black-brane parameters like horizon and charges are slowly varying was recently developed [16, 17, 18, 19], and extended to the single $U(1)$ R-charged system in ref.[20, 21, 22]. This progress is important because one can in principle go beyond the linearized approximation to arbitrary non-linear orders one desires, and some of high order transport coefficients and non-linearity seem interesting [25]. For applications to non-relativistic AdS/CFT, see ref.[26, 27, 28], and for dyonic system, see ref.[29, 30]. See also ref.[31, 32] for similar developments.

In this work, we generalize this line of development in two different ways, motivated by real QCD plasma. In QCD with several quark flavors, there is an enlarged global symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$. If one first focuses on the quark species, each quark flavor has its own conservation; in the case of three quarks u, d, s that seem relevant in the RHIC experiment, one should deal with finite temperature plasma with $U(1)^3$ global symmetry. Note that these $U(1)^3$ components are highly interacting with each other by strong interactions, and except their conservation laws one can not predict *a priori* anything about their dynamics such as diffusion coefficient etc. Because these interactions are crucially affecting the results of transport coefficients, we better work in a well-defined AdS/CFT set-up rather than working in an arbitrary unguided gravity theory. We choose to study the STU model, which is a consistent truncation of $AdS_5 \times S^5$ with $U(1)^3$ (or any Toric Sasaki-Einstein compactification) dual to $N = 4$ SYM plasma with three Cartan $U(1)$'s inside $SO(6)_R$, as a model example of multi-charged finite temperature plasma. This model has been previously studied in linear response approach in ref.[23, 24], but our framework enables one to go beyond linearized approximation. We also compute charge diffusion coefficients in the model for the first time in the literature. Although our set-up is not precisely QCD, we hope that it captures some aspect of real $U(1)^3$ dynamics in RHIC plasma.

Our second generalization is for non-Abelian $SU(2)$, although our result is valid for an arbitrary Lie group[2]. This has a clear motivation from QCD again; it mimics iso-spin $SU(2)_V$ symmetry of QCD in two-flavor approximation in a late stage of RHIC plasma. As fluctuations of charged pion density correspond to $SU(2)_V$ fluctuations, our study should be interesting in describing pion fluid at finite temperature too. For a specific model, we study a realization of $SU(2)$ gauged supergravity in Tri-Sasakian compactification of M-theory to AdS_4, corresponding to the $SU(2)_R$-symmetry sector of 3-dimensional M2-brane

[2]For study of $SU(2)$ in a different context of condensed matter system, See ref.[33, 34, 35].

plasma. However as we are more interested in higher dimensions such as 4-dimensions, we simply generalize the bosonic action to arbitrary dimensions AdS_{n+1} with $n \geq 3$ and perform analysis in complete generality of dimensions. Our analysis automatically includes the generalization of the previous single $U(1)$ case to arbitrary dimensions as well. One should however note that our $U(1)^3$ and $SU(2)$ R-symmetry in the field theory side do not come from fundamental flavors, but rather from flavors with adjoint representation. To introduce fundamental flavors, one normally needs to introduce extra branes in the set-up, whose detailed study in hydrodynamics is remained for future work.

We obtain the results at first order in derivatives, while the second order calculation is straightforward, as we will mention in the text, but extremely complicated to present. We leave its more controlled analysis to the future. We however mention that even the first order transport coefficients we obtain are quite non-trivial. In the STU model, we find three conserved currents of each $U(1)^3$ at first order in derivatives to be

$$J_I^\mu = \rho_I u^\mu - \mathcal{D}_I \left(\eta^{\mu\nu} + u^\mu u^\nu\right) D_\nu \rho_I + \zeta_I \epsilon^{\nu\rho\sigma\mu} u_\nu \partial_\rho u_\sigma + \cdots \quad , \tag{1.1}$$

where $I = 1, 2, 3$ runs for $U(1)^3$ symmetries, ρ^I is the charge density, and the diffusion coefficients $\mathcal{D}_I$ are given by

$$\mathcal{D}_I = \frac{(r_H^2 - q_I)}{2 r_H^3 H^{\frac{1}{2}}(r_H)} \quad , \tag{1.2}$$

with r_H being the horizon radius, and the relation of q_I with the energy/charge densities can be easily found in the text. The parity-violating coefficients ζ_I originated from the 5D Chern-Simons term are

$$\begin{aligned} \zeta_I &= \frac{1}{32\pi G_5}\left(C_{IJK} \frac{\sqrt{mq_J}\sqrt{mq_K}}{(r_H^2 + q_J)(r_H^2 + q_K)} \right. \\ &- \left. \frac{\sqrt{mq_I}}{3m} C_{JKL} \frac{\sqrt{mq_J}\sqrt{mq_K}\sqrt{mq_L}}{(r_H^2 + q_J)(r_H^2 + q_K)(r_H^2 + q_L)} \right) \quad . \end{aligned} \tag{1.3}$$

where C^{IJK} is the Chern-Simons coefficient. As far as we know, this is the first time in literature to have these results. For the $SU(2)$ case, our result for the first order correction to the $SU(2)$ currents in n-dimensional CFT is

$$\begin{aligned} J_\mu^{a(1)} &= -\mathcal{D}\left(\frac{\rho \cdot P_\mu^\nu (\partial_\nu \rho)}{\rho \cdot \rho} - u^\nu \partial_\nu u_\mu \right) \rho^a + \mathcal{D}_1 \epsilon^{abc} \rho^b P_\mu^\nu (\partial_\nu \rho^c) \\ &+ \mathcal{D}_2 P_\mu^\nu \left(\rho^a (\rho \cdot \partial_\nu \rho) - (\rho \cdot \rho)(\partial_\nu \rho^a) \right) \quad , \end{aligned} \tag{1.4}$$

with three diffusion coefficients

$$\mathcal{D} = \frac{1}{(n-2) r_H}\left(1 - \frac{2\vec{q}\cdot\vec{q}}{n m r_H^{n-2}}\right) = \frac{(n-2)m + 2 r_H^n}{n(n-2) m r_H} \quad ,$$

$$\mathcal{D}_1 = \frac{8\pi G_{n+1} f^{(n-2)}}{(n-1)} \quad , \quad \mathcal{D}_2 = \frac{2^{\frac{11}{2}}\pi^2 G_{n+1}^2 g^{(n-2)}}{(n-1)^{\frac{3}{2}}(n-2)^{\frac{1}{2}}} \quad . \tag{1.5}$$

The $\mathcal{D}$ is essentially the usual diffusion coefficient of Abelian nature, while the other two diffusion coefficients are due to the non-Abelian properties. The constants $f^{(n-2)}$ and $g^{(n-2)}$ are defined in the text. We hope that this structure is a useful starting point to study non-Abelian iso-spin plasma of RHIC.

2 Crash review of the method

The basic idea in ref.[16, 17] for deriving hydrodynamics from gravity is conceptually quite neat; given a black-brane solution with certain parameters such as temperature, charges etc, one simply considers slowly-varying those parameters in the solution. Note that keeping the form of the original solution while only varying the parameters inside has a physical meaning of *local* thermal equilibrium with given parameters at that point. However, the resulting configuration with varying parameters will no longer solve the equations of motion by obvious reason, and one should add corrections to the original form of the solution to satisfy the equations of motion. These corrections will clearly be sourced by the derivatives of the black-brane parameters one is slowly-varying, and one can systematically invoke derivative expansion for these corrections. After obtaining the full solution at k-'th order in derivatives, one can read off physical quantities at that order via AdS/CFT dictionary, such as energy-momentum tensor and charge currents. In principle one can go to an arbitrary order in derivative expansion systematically.

We simply illustrate the procedure for the metric and we refer to the original work of ref.[16, 17] for more complete discussion. Suppose we have a homogeneous black-brane solution with the metric $g^{(0)}_{MN}(u_\mu, m, Q_i)$ where u_μ, m and Q_i's are parameters of the solution such as 4-velocity, energy density and charge density of the plasma. Once we allow for these parameters to vary in CFT coordinate directions up to k'th derivative, one should add correction terms

$$g_{MN} = g^{(0)}_{MN}\left(u_\mu(x), m(x), Q_i(x)\right) + \sum_{i=1}^{k} g^{(i)}_{MN} \quad , \tag{2.6}$$

to solve the equations of motion, where $u_\mu(x), \ldots$ are varying in CFT coordinate x^μ up to k'th derivatives only, and $g^{(i)}_{MN}$ is a local function of derivatives at i'th order. Suppose one solved this problem up to k'th order. Then to go to the next order, one simply considers the varying parameters *in the above k'th solution* up to $(k+1)$'th order, which would not

solve the equation of motion any more due to $(k+1)$'th order in derivatives we are now considering. Because the equations of motion are solved up to k'th order already by the above, one simply needs to add $g^{(k+1)}_{MN}$ to g_{MN} that is a local function of $(k+1)$'th order in derivatives. Typically the equations of motion are second order partial differential equations, and since the variation of $g^{(k+1)}_{MN}$ itself along x^μ would be the next order to be neglected, $g^{(k+1)}_{MN}$ only depends on the holographic direction r and the *local* derivatives of parameters, without any x^μ dependence at this order. Therefore one would get a simple second order ordinary differential equation along r direction from the equations of motion,

$$L_r\left(g^{(k+1)}_{MN}\right) = S^{(k+1)}_{MN} \quad , \tag{2.7}$$

with a source $S^{(k+1)}_{MN}$ being some function of *local* derivatives at $(k+1)$'th order. Note that L_r would be universal, without being dependent on k, completely determined by the zero'th order solution, and one can obtain the source $S^{(k+1)}_{MN}$ quite straightforwardly by plugging the above k'th order solution *with* $(k+1)$*'th order derivatives of parameters* into the equation of motion, and simply gathering uncanceled $(k+1)$'th order term left. Therefore even after performing the analysis at the first order, one can find the ordinary differential operator L_r, and the subsequent higher order analysis will then become conceptually simpler. One can go on these steps inductively to arbitrary order in derivatives.

3 Hydrodynamics with $U(1)^3$: the STU model

The action of the STU model which is a sub-sector of AdS_5 gauged $N=4$ supergravity holographically dual to $N=4$ SYM theory is

$$\begin{aligned}(16\pi G_5)\mathcal{L} &= R \,+\, 2\mathcal{V}(X) \,-\, \frac{1}{2}G_{IJ}(X)(F^I)_{MN}(F^J)^{MN} - G_{IJ}(X)\partial_M X^I \partial^M X^J \\ &+ \frac{1}{24\sqrt{-g_5}}\epsilon^{MNPQR}C_{IJK}(F^I)_{MN}(F^J)_{PQ}(A^K)_R \quad ,\end{aligned} \tag{3.8}$$

where

$$\mathcal{V}(X) = 2\sum_{I=1}^{3}\frac{1}{X^I} \quad , \quad G_{IJ} = \frac{1}{2}\mathrm{diag}\left(\frac{1}{(X^I)^2}\right) \quad , \tag{3.9}$$

and C_{IJK} is totally symmetric with $C_{123}=1$. Also, X^I are not independent but constrained by

$$\frac{1}{6}C_{IJK}X^IX^JX^K = X^1X^2X^3 = 1 \quad . \tag{3.10}$$

We have put $L^4 = 4\pi g_s N l_s^4 \equiv 1$ for simplicity and in this convention, we have

$$G_5 = \frac{\pi}{2N^2} \quad , \tag{3.11}$$

where N is the rank of the gauge group. Capital letters $M, N, \cdots$ represent 5-dimensional indices, while Greek letters $\mu, \nu, \cdots$ would mean 4-dimensional indices. The equations of motion one obtains consist of the Einstein equation

$$\begin{aligned} R_{MN} &+ \left(\frac{2}{3}\mathcal{V}(X) + \frac{1}{6}G_{IJ}(X)(F^I)_{MN}(F^J)^{MN}\right) g_{MN} \\ &- G_{IJ}(X)(F^I)_{PM}(F^J)^P{}_N - G_{IJ}(X)\partial_M X^I \partial_N X^J = 0 \quad , \end{aligned} \tag{3.12}$$

the three Maxwell equations for each $I = 1, 2, 3$,

$$\nabla_N \left(G_{IJ}(X)(F^J)^{MN}\right) - \frac{1}{16\sqrt{-g_5}}\epsilon^{MNPQR}C_{IJK}(F^J)_{NP}(F^K)_{QR} = 0 \tag{3.13}$$

and the scalar field equations

$$\begin{aligned} &\left(\nabla_M \left(G_{IJ}(X)\partial^M X^J\right) + \frac{\partial \mathcal{V}(X)}{\partial X^I}\right)\frac{\delta X^I}{\delta \phi^i} \\ &-\frac{1}{2}\left(\frac{\partial G_{IJ}(X)}{\partial X^K}\right)\left(\partial_M X^I \partial^M X^J + \frac{1}{2}(F^I)_{MN}(F^J)^{MN}\right)\frac{\delta X^K}{\delta \phi^i} = 0 \quad , \end{aligned} \tag{3.14}$$

where ϕ^i $(i = 1, 2)$ are any independent parametrization of X^I's.

The black brane solution with arbitrary three charges has been known [36] and is given by [3]

$$\begin{aligned} ds^2 &= -H^{-\frac{2}{3}}(r)f(r)u_\mu u_\nu dx^\mu dx^\nu - 2H^{-\frac{1}{6}}(r)u_\mu dx^\mu dr + r^2 H^{\frac{1}{3}}(r)\left(\eta_{\mu\nu} + u_\mu u_\nu\right)dx^\mu dx^\nu \\ A^I &= \frac{\sqrt{mq_I}}{r^2 + q_I}u_\mu dx^\mu \quad , \quad X^I = \frac{H^{\frac{1}{3}}(r)}{H_I(r)} \quad , \end{aligned} \tag{3.15}$$

where

$$f(r) = -\frac{m}{r^2} + r^2 H(r) \quad , \quad H(r) = \prod_{I=1}^{3} H_I(r) \quad , \quad H_I(r) = 1 + \frac{q_I}{r^2} \quad , \tag{3.16}$$

and u_μ is the 4-velocity of the fluid with $u_\mu u^\mu = -1$. Our convention is $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +)$ and $u_\mu = (-1, 0, 0, 0)$ in the rest frame.

As we discussed in the previous section, we consider slowly varying parameters u_μ, m, and q_I up to first order, and we work in the frame where $u_\mu = (-1, 0, \cdots, 0)$ at the

[3] The gauge fields in the solution have an infinite norm at the horizon due to the diverging g^{00}, which can be remedied by going to the grand-canonical ensemble with chemical potential that can be added to the solution as a constant mode. We thank Mukund Rangamani for pointing this issue to us.

position $x^\mu = 0$ for simplicity. Once we find the solution, we can easily make the result relativistically covariant. Then at first order in derivatives, we have

$$\begin{aligned} u_\mu &= (-1, x^\mu \partial_\mu u_i) \\ m &= m^{(0)} + x^\mu \partial_\mu m \\ q_I &= q_I^{(0)} + x^\mu \partial_\mu q_I \quad , \end{aligned} \tag{3.17}$$

and the above black-brane solution will no longer be a solution with these varying parameters. To be a solution, we have to add the corrections $g^{(1)}_{MN}$, $A^{I(1)}_M$ and $X^{I(1)}$ to the zero'th order solution with varying parameters, which should be chosen to satisfy the equations of motion. These corrections will be proportional to the first derivatives of the varying parameters, and we can neglect their variations along x^μ as it would be second order, so these corrections $g^{(1)}_{MN}$, $A^{I(1)}_M$ and $X^{I(1)}$ are functions only on the r-coordinate. One can choose the gauge using coordinate re-parametrization and gauge transformations to be[4]

$$g^{(1)}_{rr} = 0 \quad , \quad g^{(1)}_{r\mu} \sim u_\mu \quad , \quad A^{I(1)}_r = 0 \quad , \quad \sum_{i=1}^{3} g^{(1)}_{ii} = 0 \quad . \tag{3.18}$$

Let us write the 0'th order metric as

$$ds^2 = -A(r)dt^2 + 2B(r)dtdr + C(r)(dx^i)^2 \quad , \tag{3.19}$$

with

$$A(r) = H^{-\frac{2}{3}}(r) f(r) \quad , \quad B(r) = H^{-\frac{1}{6}}(r) \quad , \quad C(r) = r^2 H^{\frac{1}{3}}(r) \quad . \tag{3.20}$$

Then the metric up to first order in derivatives including the correction $g^{(1)}_{MN}$ looks as

$$\begin{aligned} ds^2 &= -A(r)dt^2 + 2B(r)dtdr + C(r)(dx^i)^2 \\ &+ \left[-x^\mu (\partial_\mu A) + g^{(1)}_{tt}(r)\right] dt^2 + 2\left[x^\mu (\partial_\mu B) + g^{(1)}_{tr}(r)\right] dtdr + 2\left[-x^\mu (\partial_\mu u_i) B(r)\right] drdx^i \\ &+ 2\left[x^\mu (\partial_\mu u_i)(A(r) - C(r)) + g^{(1)}_{ti}(r)\right] dtdx^i + \left[x^\mu (\partial_\mu C)\delta_{ij} + g^{(1)}_{ij}(r)\right] dx^i dx^j \end{aligned} \tag{3.21}$$

and the gauge fields become

$$\begin{aligned} A^I &= -\frac{\sqrt{mq_I}}{r^2 + q_I} dt \\ &+ \left[-x^\mu \partial_\mu \left(\frac{\sqrt{mq_I}}{r^2 + q_I}\right) + A^{I(1)}_t(r)\right] dt + \left[x^\mu (\partial_\mu u_i) \frac{\sqrt{mq_I}}{r^2 + q_I} + A^{I(1)}_i(r)\right] dx^i , \end{aligned} \tag{3.22}$$

[4] The last gauge is different from the one in ref.[16, 17, 19, 21], but same as in ref.[18, 20]

and finally scalar fields will be

$$X^I = \frac{H^{\frac{1}{3}}(r)}{H_I(r)} + x^\mu \left(\partial_\mu X^I\right) + X^{I(1)}(r) \quad . \tag{3.23}$$

The task is to insert the above into the original equations of motion to obtain the equations for the first order corrections $g^{(1)}_{MN}$, $A^{I(1)}_M$ and $X^{I(1)}$, and then to solve them.

Because we have a spatial $SO(3)$ symmetry in our rest frame, one finds that the equations for the first order corrections decompose into different $SO(3)$ representations. After a tedious but straightforward calculation, one finds the following ordinary differential equations for $g^{(1)}_{MN}$, $A^{I(1)}_M$ and $X^{I(1)}$. The easiest one is the tensor mode, that is, traceless ij components of the Einstein equation,

$$-\frac{1}{2r}\partial_r\left(r^3 f(r)\partial_r\left(\frac{g^{(1)}_{ij}}{r^2 H^{\frac{1}{3}}(r)}\right)\right) = \frac{1}{2r}\partial_r\left(r^3 H^{\frac{1}{2}}(r)\right)\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\left(\partial_k u_k\right)\delta_{ij}\right) \quad . \tag{3.24}$$

The vector mode equations are more complicated. From ti component of the Einstein equation, one has

$$\begin{aligned} &-\frac{f(r)}{2r^3 H(r)}\partial_r\left(r^5 H(r)\partial_r\left(\frac{g^{(1)}_{ti}}{r^2 H^{\frac{1}{3}}(r)}\right)\right) - \sum_{I=1}^{3}\frac{f(r)\sqrt{mq_I}}{r^3 H(r)}\left(\partial_r A^{I(1)}_i\right) \\ &= \frac{f(r)}{H^{\frac{1}{2}}(r)}\left(\frac{2m}{r^3 f(r)} + \frac{1}{2r}\sum_{I=1}^{3}\frac{1}{H_I(r)}\right)\left(\partial_t u_i\right) + \frac{f(r)}{H^{\frac{1}{2}}(r)}\left(\frac{(\partial_i m)}{2r^3 f(r)} - P^{(1)}_i\right) \quad , \end{aligned} \tag{3.25}$$

and from the ri-component,

$$\begin{aligned} &\frac{1}{2r^3 H^{\frac{1}{2}}(r)}\partial_r\left(r^5 H(r)\partial_r\left(\frac{g^{(1)}_{ti}}{r^2 H^{\frac{1}{3}}(r)}\right)\right) + \sum_{I=1}^{3}\frac{\sqrt{mq_I}}{r^3 H^{\frac{1}{2}}(r)}\left(\partial_r A^{I(1)}_i\right) \\ &= -\frac{1}{2r}\left(\sum_{I=1}^{3}\frac{1}{H_I(r)}\right)\left(\partial_t u_i\right) + P^{(1)}_i \quad , \end{aligned} \tag{3.26}$$

where $P^{(1)}_i$ is a complex expression in terms of first order spatial derivatives in q_I's only, which is given in the Appendix. From the i-component of the Maxwell equation for each I, we have

$$\begin{aligned} &-\frac{1}{r}\partial_r\left(\frac{r f(r)\left(H_I(r)\right)^2}{H(r)}\left(\partial_r A^{I(1)}_i\right)\right) - \frac{2\sqrt{mq_I}}{r}\partial_r\left(\frac{g^{(1)}_{ti}}{r^2 H^{\frac{1}{3}}(r)}\right) \\ &= \frac{1}{r}\partial_r\left(\frac{\sqrt{mq_I}H_I(r)}{r H^{\frac{1}{2}}(r)}\left(\partial_t u_i\right)\right) + \frac{1}{r}\partial_r\left(-\frac{1}{2}C_{IJK}\frac{\sqrt{mq_J}\sqrt{mq_K}}{(r^2+q_J)(r^2+q_K)}\epsilon^{ijk}\left(\partial_j u_k\right)\right) \\ &+\frac{1}{r}\partial_r\left(\frac{1}{2r^3 H^{\frac{1}{2}}(r)\sqrt{mq_I}}\left(m(r^2-q_I)\left(\partial_i q_I\right) + q_I(r^2+q_I)\left(\partial_i m\right)\right)\right) \quad . \end{aligned} \tag{3.27}$$

The above five equations are vector mode equations. We mention that the ordinary differential operators in the left-hand side of the equations in the above and below take *integrable* forms, which is a quite non-trivial fact that has been checked via complicated algebra and educated guesses[5].

Finally, the most complicated part is the scalar mode equations under $SO(3)$. From the tt-component of Einstein equation, one has

$$
\begin{aligned}
&-\frac{f(r)}{2r^3H(r)}\partial_r\left(r^3H^{\frac{2}{3}}(r)\partial_r g_{tt}^{(1)}\right)-\frac{4}{3}\sum_{I=1}^{3}\frac{f(r)\sqrt{mq_I}}{r^3H(r)}\left(\partial_r A_t^{I(1)}\right)\\
&-\frac{f(r)}{2H^{\frac{1}{3}}(r)}\partial_r\left(H^{-\frac{2}{3}}(r)f(r)\right)\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)-\frac{8}{3}\frac{f(r)\sum_{I=1}^{3}H_I(r)}{H(r)}\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)\\
&+\frac{4}{3}\frac{f(r)}{r^4H^{\frac{4}{3}}(r)}\sum_{I=1}^{3}\left((r^2+q_I)^2+\frac{2mq_I}{r^2+q_I}\right)X^{I(1)}=S_{tt}^{(1)}\quad ,
\end{aligned}
\tag{3.28}
$$

from the tr-component,

$$
\begin{aligned}
&\frac{1}{2r^3H^{\frac{1}{2}}(r)}\partial_r\left(r^3H^{\frac{2}{3}}(r)\partial_r g_{tt}^{(1)}\right)+\frac{4}{3}\sum_{I=1}^{3}\frac{\sqrt{mq_I}}{r^3H^{\frac{1}{2}}(r)}\left(\partial_r A_t^{I(1)}\right)\\
&+\frac{1}{2}H^{\frac{1}{6}}(r)\partial_r\left(H^{-\frac{2}{3}}(r)f(r)\right)\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)+\frac{8}{3}\frac{\sum_{I=1}^{3}H_I(r)}{H^{\frac{1}{2}}(r)}\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)\\
&-\frac{4}{3}\frac{f(r)}{r^4H^{\frac{5}{6}}(r)}\sum_{I=1}^{3}\left((r^2+q_I)^2+\frac{2mq_I}{r^2+q_I}\right)X^{I(1)}=S_{tr}^{(1)}\quad ,
\end{aligned}
\tag{3.29}
$$

the rr-component looks as

$$
\partial_r\left(\log\left(r^3H^{\frac{1}{2}}(r)\right)\right)\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)-\sum_{I=1}^{3}\partial_r\left(\log\left(\frac{H^{\frac{1}{3}}(r)}{H_I(r)}\right)\right)\partial_r\left(\frac{H_I(r)}{H^{\frac{1}{3}}(r)}X^{I(1)}\right)=0,
\tag{3.30}
$$

and the last scalar mode equation from the Einstein equation is the trace part, that is $\sum_{i=1}^{3}(ii)$,

$$
\begin{aligned}
&\frac{3}{2r}\partial_r\left(rH^{\frac{1}{3}}(r)\partial_r\left(r^2H^{\frac{1}{3}}(r)\right)g_{tt}^{(1)}\right)-\frac{2}{r}\sum_{I=1}^{3}\sqrt{mq_I}\left(\partial_r A_t^{I(1)}\right)\\
&+\frac{3}{2}\frac{\partial_r\left(r^2H^{\frac{1}{3}}(r)\right)f(r)}{H^{\frac{1}{3}}(r)}\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)+8r^2\left(\sum_{I=1}^{3}H_I(r)\right)\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)\\
&-\frac{4}{r^2H^{\frac{1}{3}}(r)}\sum_{I=1}^{3}\left((r^2+q_I)^2-\frac{mq_I}{r^2+q_I}\right)X^{I(1)}=\sum_{i=1}^{3}S_{ii}^{(1)}\quad ,
\end{aligned}
\tag{3.31}
$$

[5]In verifying these, we sometimes used Mathematica for basic algebra manipulations.

where $S_{tt}^{(1)}$, $S_{tr}^{(1)}$, and $\sum_{i=1}^{3} S_{ii}^{(1)}$ are source terms proportional to space-time derivatives of black-brane parameters, which are given in the Appendix.

One obtains more scalar mode equations from the Maxwell equations. From the t-part of the Maxwell equations for each I, one gets

$$\frac{2f(r)\sqrt{mq_I}}{r^3H(r)}\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)-\frac{f(r)}{r^3H(r)}\partial_r\left(r^3H_I^2(r)\partial_rA_t^{I(1)}\right)$$
$$+\frac{4f(r)\sqrt{mq_I}}{r^3H(r)}\partial_r\left(\frac{H_I(r)}{H^{\frac{1}{3}}(r)}X^{I(1)}\right)=\frac{2}{r^3H^{\frac{1}{2}}(r)}\Big(\partial_t\left(\sqrt{mq_I}\right)+\sqrt{mq_I}\left(\partial_iu_i\right)\Big) \quad ,(3.32)$$

and from the r-component,

$$\frac{-2\sqrt{mq_I}}{r^3H^{\frac{1}{2}}(r)}\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)+\frac{1}{r^3H^{\frac{1}{2}}(r)}\partial_r\left(r^3H_I^2(r)\partial_rA_t^{I(1)}\right)$$
$$-\frac{4\sqrt{mq_I}}{r^3H^{\frac{1}{2}}(r)}\partial_r\left(\frac{H_I(r)}{H^{\frac{1}{3}}(r)}X^{I(1)}\right)=0 \quad . \qquad (3.33)$$

Lastly, we present the scalar field equations. For this purpose, we choose X^1 and X^2 as independent variables with $X^3=\frac{1}{X^1X^2}$. We have

$$-\frac{1}{r^3H^{\frac{1}{3}}(r)}\partial_r\left(r^3H^{\frac{2}{3}}(r)\partial_r\left(\log\left(\frac{H^{\frac{1}{2}}(r)}{H_1(r)H_2^{\frac{1}{2}}(r)}\right)\right)g_{tt}^{(1)}\right)$$
$$-\frac{2}{r^3H^{\frac{1}{3}}(r)}\left(\sqrt{mq_1}\left(\partial_rA_t^{1(1)}\right)-\sqrt{mq_3}\left(\partial_rA_t^{3(1)}\right)\right)$$
$$-\frac{f(r)}{H^{\frac{1}{3}}(r)}\left(\log\left(\frac{H^{\frac{1}{2}}(r)}{H_1(r)H_2^{\frac{1}{2}}(r)}\right)\right)\partial_r\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)-\frac{4(q_1-q_3)}{r^2H^{\frac{1}{3}}(r)}\left(H^{\frac{1}{6}}(r)g_{tr}^{(1)}\right)$$
$$+\frac{1}{r^3H^{\frac{1}{3}}(r)}\partial_r\left(r^3f(r)\partial_r\left(\frac{H_1(r)}{H^{\frac{1}{3}}(r)}X^{1(1)}+\frac{H_2(r)}{2H^{\frac{1}{3}}(r)}X^{2(1)}\right)\right)$$
$$+\frac{2}{r^2H^{\frac{1}{3}}(r)}\left((r^2+q_1)+\frac{2mq_1}{(r^2+q_1)^2}\right)\left(\frac{H_1(r)}{H^{\frac{1}{3}}(r)}X^{1(1)}\right)$$
$$+\frac{2}{r^2H^{\frac{1}{3}}(r)}\left((r^2+q_3)+\frac{2mq_3}{(r^2+q_3)^2}\right)\left(\frac{H_1(r)}{H^{\frac{1}{3}}(r)}X^{1(1)}+\frac{H_2(r)}{H^{\frac{1}{3}}(r)}X^{2(1)}\right)=S^{1(1)}, \qquad (3.34)$$

and the similar equation with X^1 and X^2 interchanged. The source terms $S^{1(1)}$ and $S^{2(1)}$ can again be found in the Appendix. In the Appendix, we also sketch our method of computations for deriving these equations, the main task being to obtain the variation of Ricci tensor up to first order.

The main point in this heavy endeavor is in fact to solve the above equations to find the first order transport coefficients; luckily enough, we are able to solve the above equations *in explicit integral forms.*

3.1 The solution

We first observe that some combinations of the above equations are in fact *constraints* on the space-time derivatives of the black-brane parameters; they can't be arbitrary but have to be consistent with the conservations laws in the CFT side, such as energy-momentum and current conservations. In other words, bulk equations of motion *include* the conservation laws in the CFT side. Writing the Einstein equation as E_{MN} and the Maxwell equations as M^I_M, one has three kinds of constraints equations,

$$
\begin{aligned}
0 &= g^{rt}E_{tt} + g^{rr}E_{rt} = -\left(H^{\frac{1}{6}}(r)S^{(1)}_{tt} + H^{-\frac{1}{3}}(r)f(r)S^{(1)}_{tr}\right) \\
&= \frac{-1}{2r^9 H^{\frac{3}{2}}(r)}\Bigg(\left(2mq_1q_2q_3\left(\partial_i u_i\right) + m\partial_t\left(q_1q_2q_3\right)\right) \\
&- \left(\left(\partial_t m\right)\left(q_1q_2+q_2q_3+q_3q_1\right) - m\partial_t\left(q_1q_2+q_2q_3+q_3q_1\right)\right) r^2 \\
&- \left(2m\left(\partial_i u_i\right)\left(q_1+q_2+q_3\right) + 2\left(\partial_t m\right)\left(q_1+q_2+q_3\right) - m\partial_t\left(q_1+q_2+q_3\right)\right) r^4 \\
&- \left(4m\left(\partial_i u_i\right) + 3\left(\partial_t m\right)\right) r^6 \Bigg) \quad , \qquad (3.35)\\
0 &= g^{rt}M^I_t + g^{rr}M^I_r = \frac{-2}{r^3 H^{\frac{1}{3}}(r)}\left(\partial_t\left(\sqrt{mq_I}\right) + \sqrt{mq_I}\left(\partial_i u_i\right)\right) \quad , \qquad (3.36)\\
0 &= g^{rt}E_{ti} + g^{rr}E_{ri} = \frac{-1}{r^3 H^{\frac{1}{3}}(r)}\left(\frac{1}{2}\left(\partial_i m\right) + 2m\left(\partial_t u_i\right)\right) \quad . \qquad (3.37)
\end{aligned}
$$

Note that for the above combinations of equations of motions, the radial differential operators cancel with each other to leave the above algebraic constraints. The constraints are uniquely solved by

$$
\begin{aligned}
\left(\partial_t m\right) &= -\frac{4}{3}m\left(\partial_i u_i\right) \quad , \quad \left(\partial_i m\right) = -4m\left(\partial_t u_i\right) \quad , \\
\left(\partial_t q_I\right) &= -\frac{2}{3}q_I\left(\partial_i u_i\right) \quad \text{or equivalently,} \quad \partial_t\left(\sqrt{mq_I}\right) = -\sqrt{mq_I}\left(\partial_i u_i\right) \quad , \qquad (3.38)
\end{aligned}
$$

where the first two equations imply the *zero'th order* energy momentum conservation, and the last equation is the conservation of $U(1)^3$ global symmetry currents. Indeed, applying AdS/CFT dictionary to our zero'th order black-brane solution, we have

$$
\begin{aligned}
T^{\mu\nu(0)} &= \frac{m}{16\pi G_5}\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) \equiv p\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) \quad , \\
J^{\mu(0)}_I &= \frac{\sqrt{mq_I}}{8\pi G_5}u^\mu \equiv \rho_I u^\mu \quad , \qquad (3.39)
\end{aligned}
$$

whose conservation laws in our rest-frame $u_\mu = (-1,0,0,0)$ are nothing but (3.38). One generically obtains conservation laws of $(k-1)$'th order from the k'th order equations of motion.

We next solve the remaining dynamical equations. It is easiest to solve the tensor mode equation (3.24). Integrating it gives us

$$g_{ij}^{(1)}(r) = r^2 H^{\frac{1}{3}}(r)\left(-2\sigma_{ij}\int_\infty^r dr'\,\frac{H^{\frac{1}{2}}(r')}{f(r')} + C_{ij}\int_\infty^r dr'\,\frac{1}{r'^3 f(r')} + C'_{ij}\right) \quad , \tag{3.40}$$

with some constants C_{ij} and C'_{ij}, and we define

$$\sigma_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\left(\partial_k u_k\right)\delta_{ij}\right) \quad . \tag{3.41}$$

One has to put $C'_{ij} = 0$ as it is a non-normalizable mode. The C_{ij} is uniquely determined to give a regular solution at the horizon $r = r_H$ where $f(r_H) = 0$; note that the two integrals in the above logarithmically diverge near $r = r_H$, and having a cancelation between the two for a finite result uniquely fixes

$$C_{ij} = 2r_H^3 H^{\frac{1}{2}}(r_H)\sigma_{ij} \quad . \tag{3.42}$$

The vector mode equations (3.25), (3.26), and (3.27) are harder. As we already solved one constraint equation from (3.25) and (3.26), we need to solve only (3.26) and (3.27). Let us first integrate (3.27) once, which gives us

$$\begin{aligned}
&\frac{rf(r)\left(H_I(r)\right)^2}{H(r)}\left(\partial_r A_i^{I(1)}\right) + 2\sqrt{mq_I}\left(\frac{g_{ti}^{(1)}}{r^2 H^{\frac{1}{3}}(r)}\right) \\
= \quad & -\frac{\sqrt{mq_I}H_I(r)}{rH^{\frac{1}{2}}(r)}\left(\partial_t u_i\right) + \frac{1}{2}C_{IJK}\frac{\sqrt{mq_J}\sqrt{mq_K}}{(r^2+q_J)(r^2+q_K)}\epsilon^{ijk}\left(\partial_j u_k\right) \\
- \quad & \frac{1}{2r^3 H^{\frac{1}{2}}(r)\sqrt{mq_I}}\Big(m(r^2-q_I)\left(\partial_i q_I\right) + q_I(r^2+q_I)\left(\partial_i m\right)\Big) + C_i^I \equiv Q_i^{I(1)}(r) + C_i^I \quad ,
\end{aligned} \tag{3.43}$$

with some integration constants C_i^I, and then consider the horizon $r = r_H$ where $f(r_H) = 0$. Imposing a regularity on $A_i^{I(1)}$ and $g_{ti}^{(1)}$ at $r = r_H$, the first term drops and we have

$$\left(\frac{g_{ti}^{(1)}(r_H)}{r_H^2 H^{\frac{1}{3}}(r_H)}\right) = \frac{1}{2\sqrt{mq_I}}\left(Q_i^{I(1)}(r_H) + C_i^I\right) \quad . \tag{3.44}$$

The point is that the left-hand side is independent of I, so that C_i^I can not be arbitrary but has to take a form

$$C_i^I = -Q_i^{I(1)}(r_H) + \frac{2\sqrt{mq_I}}{r_H^2 H^{\frac{1}{3}}(r_H)}C_i \quad , \tag{3.45}$$

with only one degree of freedom C_i, which is nothing but the value of $g_{ti}^{(1)}(r_H)$ at the horizon. The next step is to use the above (3.43) to replace $\left(\partial_r A_i^{I(1)}\right)$ in the equation (3.26) to get a second order differential equation for $g_{ti}^{(1)}$ only;

$$\begin{aligned} & \partial_r\left(r^5 H(r)\partial_r\left(\frac{g_{ti}^{(1)}}{r^2 H^{\frac{1}{3}}(r)}\right)\right)-\left(\sum_{I=1}^{3}\frac{4mq_I H(r)}{rf(r)\left(H_I(r)\right)^2}\right)\left(\frac{g_{ti}^{(1)}}{r^2 H^{\frac{1}{3}}(r)}\right) \qquad (3.46)\\ = & \sum_{I=1}^{3}\frac{-2\sqrt{mq_I}H(r)}{rf(r)\left(H_I(r)\right)^2}\left(Q_i^{I(1)}(r)-Q_i^{I(1)}(r_H)+\frac{2\sqrt{mq_I}}{r_H^2 H^{\frac{1}{3}}(r_H)}C_i\right)\\ - & r^2 H^{\frac{1}{2}}(r)\left(\sum_{I=1}^{3}\frac{1}{H_I(r)}\right)(\partial_t u_i)+2r^3 H^{\frac{1}{2}}(r)P_i^{(1)}(r)\,. \end{aligned}$$

To our surprise, the second order differential operator in the left-hand side is in fact *integrable*, that is, the left-hand side can be transformed into

$$\frac{r^2 H(r)}{f(r)}\partial_r\left(\frac{r\left(f(r)\right)^2}{H(r)}\partial_r\left(\frac{H^{\frac{2}{3}}(r)}{f(r)}g_{ti}^{(1)}\right)\right) \quad . \qquad (3.47)$$

The way we have found this is the following. We start from the Ansatz for an integrable form,

$$\frac{1}{P}\partial_r\left(Q\partial_r\left(S\cdot\right)\right)=\frac{1}{P}\left(QS\partial_r^2\cdot+\left(\partial_r\left(QS\right)+Q\left(\partial_r S\right)\right)\partial_r\cdot+\partial_r\left(Q\partial_r S\right)\cdot\right) \quad , \qquad (3.48)$$

and comparing with the left-hand side of (3.46), one gets

$$\frac{1}{P}QS=r^5 H(r) \quad , \quad \frac{1}{P}\left(\partial_r\left(QS\right)+Q\left(\partial_r S\right)\right)=\partial_r\left(r^5 H(r)\right)\,, \qquad (3.49)$$

$$\frac{1}{P}\partial_r\left(Q\partial_r S\right)=-\sum_{I=1}^{3}\frac{4mq_I H(r)}{rf(r)\left(H_I(r)\right)^2} \quad . \qquad (3.50)$$

One can easily remove Q and S in terms of P to get a differential equation for P, which turns out to be the same differential equation (3.46), but now without the source term in the right-hand side; that is, P is a homogeneous solution of the differential operator in (3.46). Luckily, we know one way to *generate* a homogeneous solution without source terms; recall that any coordinate re-parametrization must correspond to a homogeneous solution of the problem. For our purpose, the following infinitesimal coordinate transformation

$$dt\to dt-\epsilon dx^i \quad , \quad dx^i\to dx^i+\epsilon\frac{1}{r^2 H^{\frac{1}{2}}(r)}dr \quad , \qquad (3.51)$$

generates one homogeneous solution for $g_{ti}^{(1)}$, which is $H^{-\frac{2}{3}}(r)f(r)$, and from this one can let P be

$$P=\frac{H^{-\frac{2}{3}}(r)f(r)}{r^2 H^{\frac{1}{3}}(r)}=\frac{f(r)}{r^2 H(r)} \quad . \qquad (3.52)$$

Once P is found, it is straightforward to find Q and S from the above equation to have the above integrable form of the differential operator.

Solving (3.46) then by integrating it once, we have

$$\begin{aligned}
&\frac{r\,(f(r))^2}{H(r)}\partial_r\left(\frac{H^{\frac{2}{3}}(r)}{f(r)}g_{ti}^{(1)}\right) \\
&= \int_\infty^r dr'\left(\sum_{I=1}^{3}\frac{-2\sqrt{mq_I}}{(r')^3\,(H_I(r'))^2}\left(Q_i^{I(1)}(r') - Q_i^{I(1)}(r_H) + \frac{2\sqrt{mq_I}}{r_H^2 H^{\frac{1}{3}}(r_H)}C_i\right)\right. \\
-&\left.\frac{f(r')}{H^{\frac{1}{2}}(r')}\left(\sum_{I=1}^{3}\frac{1}{H_I(r')}\right)(\partial_t u_i) + \frac{2r' f(r')}{H^{\frac{1}{2}}(r')}P_i^{(1)}(r')\right) + C_i' \\
\equiv&\int_\infty^r dr'\, I(r') + C_i',
\end{aligned} \tag{3.53}$$

with an integration constant C_i', which can be fixed by considering the behavior at the horizon. Note that $f(r)$ has an expansion near $r = r_H$ as

$$f(r) = f'(r_H)(r - r_H) + \frac{1}{2}f''(r_H)(r - r_H)^2 + \cdots \quad , \tag{3.54}$$

with $f'(r_H) > 0$, and the left-hand side in the above (3.53) takes a limit as $r \to r_H$,

$$\frac{r\,(f(r))^2}{H(r)}\partial_r\left(\frac{H^{\frac{2}{3}}(r)}{f(r)}g_{ti}^{(1)}\right) \to -\frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}g_{ti}^{(1)}(r_H) = -\frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}C_i \quad , \tag{3.55}$$

where we have used the fact that $g_{ti}^{(1)}(r_H) = C_i$ previously. This fixes C_i' to be

$$C_i' = -\int_\infty^{r_H} dr' I(r') - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}C_i \quad , \tag{3.56}$$

where $I(r')$ is the same integrand in (3.53), and one can rewrite (3.53) as

$$\frac{r\,(f(r))^2}{H(r)}\partial_r\left(\frac{H^{\frac{2}{3}}(r)}{f(r)}g_{ti}^{(1)}\right) = \int_{r_H}^r dr'\, I(r') - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}C_i \quad . \tag{3.57}$$

We then integrate the above once more to have

$$g_{ti}^{(1)}(r) \;=\; \frac{f(r)}{H^{\frac{2}{3}}(r)}\int_\infty^r dr'\,\frac{H(r')}{r'\,(f(r'))^2}\left(\int_{r_H}^{r'} dr''\, I(r'') - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}C_i\right) + \frac{f(r)}{H^{\frac{2}{3}}(r)}C_i'' ,$$

where one needs to put the integration constant C_i'' to be zero as it corresponds precisely to the coordinate re-parametrization that we have used to get a homogeneous solution.

At this point, the only uncertainty we have to fix is the constant C_i that appears both in $I(r)$ and the above result. It might seem that it can be fixed for the $g_{ti}^{(1)}$ to have a

regular *derivative* at the horizon $r = r_H$; observe that the above result for $g^{(1)}_{ti}$ already has a finite *value* at the horizon irrespective of the constant C_i. Suppose that the integrand in the above equation has an expansion near $r = r_H$,

$$\frac{H(r')}{r'\left(f(r')\right)^2}\left(\int_{r_H}^{r'} dr''\, I(r'') - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}C_i\right) \sim \frac{a}{(r'-r_H)^2} + \frac{b}{(r'-r_H)} + \cdots \quad , \tag{3.58}$$

then the first piece is harmless after integration as it cancels with $f(r)$ in front, while the second piece will result in

$$g^{(1)}_{ti} \sim (r-r_H)\log(r-r_H) + \cdots \quad , \tag{3.59}$$

which has a divergent *radial derivative* that would signal divergent curvature tensors. Explicitly, one finds that the Ricci tensor R_{ri} diverges with this. Therefore, we may have to choose right C_i to make sure that $b = 0$ in the near horizon expansion. Explicitly, one has

$$b = \frac{H(r_H)}{r_H\left(f'(r_H)\right)^2}\left(I(r_H) - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)}\left(\frac{H'(r_H)}{H(r_H)} - \frac{1}{r_H} - \frac{f''(r_H)}{f'(r_H)}\right)C_i\right) \quad , \tag{3.60}$$

and moreover one finds from (3.53) using $f(r_H) = 0$ that

$$I(r_H) = \sum_{I=1}^{3} \frac{-4mq_I}{r_H^5 H^{\frac{1}{3}}(r_H)\left(H_I(r_H)\right)^2} C_i \quad , \tag{3.61}$$

so that b is in fact proportional to $C_i = g^{(1)}_{ti}(r_H)$, and it appears that we may have to put it zero for regularity. However, an explicit computation shows that the coefficient in front of C_i in fact vanishes identically, and the geometry is smooth for *any* value of C_i. We need an extra input to fix the constant C_i.

The answer to this puzzle lies in the *frame choice*, more concretely, the choice of either Landau frame or Eckart frame. In our work, we choose the Landau frame which states that

$$u_\mu T^{\mu\nu} = -\epsilon u^\nu \quad , \tag{3.62}$$

and in particular, $T^{ti} = 0$ must hold in our local rest frame. In holographic renormalization that we will discuss in more detail in the next section, this condition gives us the constraint that the coefficient of $\frac{1}{r^2}$ in near boundary expansion of $g^{(1)}_{ti}(r)$ should vanish, because it is precisely proportional to the first order correction to T^{ti}. One can easily check that this indeed fixes our integration constant C_i uniquely. We have then completely solved

for $g_{ti}^{(1)}(r)$, as given above with C_i determined. Our explicit calculations give us

$$\begin{aligned} C_i &= \frac{r_H^2 H^{\frac{1}{3}}(r_H)}{4m}\Bigg(\sum_{I=1}^{3}\Bigg(\frac{4m}{\sqrt{mq_I}}\Bigg(\mathcal{D}_I - \frac{(r_H^2 - q_I)}{2r_H^2 H^{\frac{1}{2}}(r_H)}\Bigg)(\partial_i\sqrt{mq_I})\Bigg) \\ &+ \frac{\sqrt{m}}{r_H}\Bigg(4r_H^2 + 3\sum_{J=1}^{3} q_J\Bigg)(\partial_t u_i) \\ &+ \frac{1}{3}C_{IJK}\frac{\sqrt{mq_I}\sqrt{mq_J}\sqrt{mq_K}}{(r_H^2+q_I)(r_H^2+q_J)(r_H^2+q_K)}\epsilon^{ijk}(\partial_j u_k)\Bigg) \quad , \end{aligned} \tag{3.63}$$

where $\mathcal{D}_I$ is given in the next section.

Once we have the solution for $g_{ti}^{(1)}(r)$, one simply plugs it into the equation (3.43) to solve for $A_i^{I(1)}$, whose integration gives us

$$A_i^{I(1)} = \int_\infty^r dr' \, \frac{H(r')}{r' f(r') \left(H_I(r')\right)^2}\left(Q_i^{I(1)}(r') - Q_i^{I(1)}(r_H) - 2\sqrt{mq_I}\left(\frac{g_{ti}^{(1)}(r')}{(r')^2 H^{\frac{1}{3}}(r')} - \frac{C_i}{r_H^2 H^{\frac{1}{3}}(r_H)}\right)\right), \tag{3.64}$$

where we have chosen the integration constant to remove non-normalizable modes. This completes our long solution for the vector modes under $SO(3)$.

Finally we come to the solution of the scalar modes under $SO(3)$. Although it seems difficult to our eyes to systematically solve these equations, we are lucky to be able to solve them by an educated guess from the previous analysis in ref.[16, 21]; in fact, for the case of single $U(1)$ R-charged hydrodynamics, most of the scalar modes under $SO(3)$ turn out to be zero, *except* $g_{tt}^{(1)}$. Assuming the same feature, one easily finds that

$$g_{tt}^{(1)} = \frac{2}{3} r H^{-\frac{1}{6}}(r)\,(\partial_i u_i) \quad , \tag{3.65}$$

indeed solves all the scalar mode equations of motion in the previous section[6]. As the solution is expected to be unique up to trivial coordinate re-parametrizations, we conclude that the above $g_{tt}^{(1)}$ with

$$g_{tr}^{(1)} = A_t^{I(1)} = X^{I(1)} = 0 \quad , \tag{3.66}$$

is the solution of the scalar modes under $SO(3)$.

3.2 The first order transport coefficients

It is a standard AdS/CFT procedure to obtain the first order corrections to the CFT energy-momentum tensor and the $U(1)^3$ symmetry currents from the results in the previous section. We first discuss the energy-momentum tensor. One way to compute the CFT

[6] We used Mathematica for basic algebra manipulations in showing this.

energy-momentum tensor is to rewrite the full first-order metric in the Fefferman-Graham coordinate,[7] where

$$ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2 g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \quad , \tag{3.67}$$

and to read off the coefficient of the large ρ-expansion of $g_{\mu\nu}(\rho, x)$,

$$g_{\mu\nu}(\rho, x) \sim \eta_{\mu\nu} + \cdots + \frac{g^{(4)}_{\mu\nu}(x)}{\rho^4} + \cdots \quad . \tag{3.68}$$

The holographic renormalization procedure [37] would then give us

$$T_{\mu\nu} = \frac{1}{4\pi G_5} g^{(4)}_{\mu\nu}(x) \quad . \tag{3.69}$$

However, if one naively applies this to the zero'th order black-brane solution (3.15), one *does not* find the previously quoted energy-momentum tensor in (3.39),

$$T^{\mu\nu(0)} = \frac{m}{16\pi G_5} \left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) \quad . \tag{3.70}$$

This is due to a subtlety in the scalar fields sector of the STU model; the scalar fields sector provides the cosmological constant at its vacuum, and hence the boundary counter-term that one adds in the holographic renormalization involves a non-trivial potential term of the scalar fields X^I [38]. Because X^I in the solution has a non-trivial profile, it turns out that this counter-term gives an additional contribution to the energy-momentum tensor other than (3.69). The careful analysis including this subtlety was carried out in ref.[23] to find the above correct answer.

For our purpose to find the first order correction $T^{\mu\nu(1)}$, we are however in a lucky situation. Because the first order corrections to the scalar fields $X^{I(1)}$ vanish, there wouldn't be any first order contributions from the scalar fields sector to the energy-momentum tensor, and we can safely use (3.69) for the first order corrections to the energy-momentum tensor. Be warned that this may not be true in higher orders. A direct expansion of our first-order solutions in the Fefferman-Graham coordinate ρ which are related to our r variable by

$$r = \rho + \frac{(\partial_i u_i)}{3} - \frac{\left(\sum_I q_I\right)}{6\rho} + \frac{(\partial_i u_i)\left(\sum_I q_I\right)}{54\rho^2} + \cdots \quad , \tag{3.71}$$

one finds that the only non-vanishing first-order correction to the energy-momentum comes from $g^{(1)}_{ij}$, and is given by

$$T^{(1)}_{ij} = -2\frac{r_H^3 H^{\frac{1}{2}}(r_H)}{16\pi G_5}\sigma_{ij} = -2\frac{s}{4\pi}\sigma_{ij} \equiv -2\eta\sigma_{ij} \quad , \tag{3.72}$$

[7] It is also possible to get the energy-momentum tensor directly in the Eddington-Finkelstein coordinate, but the end results should be same in the first order in derivatives.

where we have used the fact that the horizon area per unit CFT volume is given by $r_H^3 H^{\frac{1}{2}}(r_H)$, and the entropy density from the Bekenstein-Hawking formula gives

$$s = \frac{r_H^3 H^{\frac{1}{2}}(r_H)}{4G_5} \quad . \tag{3.73}$$

The last equality is simply the definition of shear viscosity η, and one recovers the famous ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad , \tag{3.74}$$

in the STU model. It is not hard to make the previous result in a manifestly covariant standard form away from our static frame $u_\mu = (-1, 0, 0, 0)$;

$$T^{\mu\nu} = p\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu} + \cdots \quad , \tag{3.75}$$

with

$$\sigma^{\mu\nu} = \frac{1}{2}P^{\mu\alpha}P^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) - \frac{1}{3}P^{\mu\nu}\left(\partial_\alpha u^\alpha\right) \quad , \tag{3.76}$$

where $P^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$ is the projection to the transverse components to u^μ.

It is also possible to write the first-order corrected metric in the covariant form,

$$\begin{aligned} ds^2 &= -H^{2/3}(r)f(r)u_\mu u_\nu dx^\mu dx^\nu - 2H^{1/6}u_\mu dx^\mu dr + r^2 H^{1/3}(r)P_{\mu\nu}dx^\mu dx^\nu \\ &+ \frac{2}{3}rH^{-1/6}(r)(\partial_\rho u_\rho)u_\mu u_\nu dx^\mu dx^\nu \\ &- 2\frac{f(r)}{H^{2/3}(r)}u_\mu\left(\int_\infty^r dr'\frac{H(r')}{r'f^2(r')}\left(\int_{r_H}^{r'} dr'' I_\nu^{(1)}(r'') - \frac{r_H f'(r_H)}{H^{1/3}(r_H)}C_\mu\right)\right)dx^\mu dx^\nu \\ &+ 2r_H^3 H^{1/2}(r_H)r^2 H^{1/3}(r)\sigma_{\mu\nu}\left(\int_\infty^r dr'\frac{1}{f(r')}\left(\frac{1}{r'^3} - \frac{1}{r_H^3}\frac{H^{1/2}(r')}{H^{1/2}(r_H)}\right)\right)dx^\mu dx^\nu \quad . \end{aligned} \tag{3.77}$$

Here, the covariant first-order radial function $I_\mu^{(1)}(r)$ is defined as

$$\begin{aligned} I_\mu^{(1)}(r) &= -2\sum_{I=1}^{3}\frac{\sqrt{mq_I}}{(r)^3 H_I^2(r)}\left(Q_\mu^{I(1)}(r) - Q_\mu^{I(1)}(r_H)\right) - \frac{f(r)}{H^{1/2}(r)}u^\nu\partial_\nu u_\mu\sum_{I=1}^{3}\frac{1}{H_I(r)} \\ &+ 2\frac{rf(r)}{H^{1/2}(r)}P_\mu^{(1)}(r) \quad , \end{aligned} \tag{3.78}$$

where the first-order functions $Q_\mu^{I(1)}(r)$ and $P_\mu^{I(1)}(r)$ are defined respectively as

$$\begin{aligned} Q_\mu^{I(1)}(r) &= -\frac{\sqrt{mq_I}H_I(r)}{rH^{1/2}(r)}P_\mu^\nu u^\alpha\partial_\alpha u_\nu + \frac{1}{2}C^{IJK}\frac{\sqrt{mq_J}\sqrt{mq_K}}{(r^2+q_J)(r^2+q_K)}\epsilon^{\nu\rho\sigma\mu}u_\nu\partial_\rho u_\sigma \\ &- \frac{1}{2r^3 H^{1/2}(r)\sqrt{mq_I}}P_\mu^\nu\left(m(r^2 - q_I)\partial_\nu q_I + q_I(r^2+q_I)\partial_\nu m\right) \quad , \\ P_\mu^{(1)}(r) &= \frac{1}{4r^3 H(r)}P_\mu^\nu\left(H(r)\sum_{I=1}^{3}\frac{\partial_\nu q_I}{H_I^2(r)} - \sum_{I=1}^{3}H_I(r)\sum_{J=1}^{3}\partial_\nu q_J + \sum_{I=1}^{3}H_I(r)\partial_\nu q_I\right) \end{aligned} \tag{3.79}$$

The C_μ is the covariantized form of C_i determined before.

We next obtain the first order corrections to the $U(1)^3$ currents, which haven't been computed before in the literature, and would be the first non-trivial results in this work. The standard AdS/CFT formula

$$J_I^\mu = \lim_{\rho\to\infty} \frac{\rho^2}{8\pi G_5}\eta^{\mu\nu} A_\nu^I(\rho) \quad , \tag{3.80}$$

works fine here, and it is easy to get the covariantized first order correction

$$J_\mu^{I(1)} = \frac{1}{16\pi G_5}\left(Q_\mu^{I(1)}(r_H) - \frac{2\sqrt{mq_I}}{r_H^2 H^{\frac{1}{3}}(r_H)} C_\mu \right) \quad , \tag{3.81}$$

where $Q_\mu^{I(1)}(r)$ is defined in the above. Using the covariant version of the conservation (3.38),

$$P_\mu^\nu \partial_\nu m = -4m P_\mu^\nu \left(u^\alpha \partial_\alpha u_\nu\right) \quad , \tag{3.82}$$

and the expression for the density ρ_I in (3.39), one can rewrite the result in a more suggestive form, up to first order in derivatives

$$J_I^\mu = \rho_I u^\mu - \mathcal{D}_I P^{\mu\nu} D_\nu \rho_I + \zeta_I \epsilon^{\nu\rho\sigma\mu} u_\nu \partial_\rho u_\sigma + \cdots \quad , \tag{3.83}$$

where the diffusion coefficient $\mathcal{D}_I$ is given by

$$\mathcal{D}_I = \frac{(r_H^2 - q_I)}{2r_H^3 H^{\frac{1}{2}}(r_H)} \quad , \tag{3.84}$$

and the parity-violating coefficient ζ_I (originated from the 5D Chern-Simons term) is

$$\begin{aligned} \zeta_I \;&=\; \frac{1}{32\pi G_5}\left(C_{IJK} \frac{\sqrt{mq_J}\sqrt{mq_K}}{(r_H^2+q_J)(r_H^2+q_K)} \right. \\ &\left. - \;\frac{\sqrt{mq_I}}{3m} C_{JKL} \frac{\sqrt{mq_J}\sqrt{mq_K}\sqrt{mq_L}}{(r_H^2+q_J)(r_H^2+q_K)(r_H^2+q_L)} \right) \quad . \end{aligned} \tag{3.85}$$

We finish this section by presenting the full covariant form of the gauge potential up to first order in derivatives;

$$\begin{aligned} A^I \;&=\; \left(\frac{\sqrt{mq_I}}{r^2+q_I} u_\mu + \int_\infty^r dr' \frac{H(r)}{r' f(r') H_I^2(r')} \left(Q_\mu^I(r') - Q_\mu^I(r_H) + \frac{2\sqrt{mq_I}}{r_H^2 H^{\frac{1}{3}}(r_H)} C_\mu \right) \right. \\ &\left. - \; 2\sqrt{mq_I} \int_\infty^r dr' \frac{1}{(r')^3 H_I^2(r')} \int_\infty^{r'} dr'' \frac{H(r'')}{r'' f^2(r'')} \left(\int_{r_H}^{r''} dr''' I_\mu(r''') - \frac{r_H f'(r_H)}{H^{\frac{1}{3}}(r_H)} C_\mu \right) \right) dx^\mu \quad , \end{aligned} \tag{3.86}$$

where appropriate functions are defined previously.

4 Hydrodynamics with $SU(2)$ in arbitrary dimensions

Our next subject is to consider non-Abelian symmetry dynamics in hot hydrodynamic plasmas, motivated by the iso-spin $SU(2)_I$ dynamics in the QCD plasma. Although one can embed our bosonic action into the well-defined AdS_4/CFT_3 set-up of the Tri-Sasakian compactification of M-theory to AdS_4, we perform our analysis in arbitrary dimensions envisioning that any system with non-Abelian symmetry would be described, at least approximately, by our model. In fact, the action we study has the simplest form one can imagine with gravity and gauge fields in AdS. However, in general dimensions other than $n = 4$ the connection to real QCD will no longer be our main motivation.

We will consider $(n+1)$-dimensional gravity corresponding to n-dimensional CFT. We restrict ourselves to the cases of $n \geq 3$ only, as the $n = 2$ case seems peculiar in our results. Our action contains gravity with $SU(2)$ gauge fields[8]

$$\mathcal{L} = \frac{1}{16\pi G_{n+1}} \Big(R + n(n-1) - F^a_{MN} F^{aMN} \Big) \quad , \tag{4.87}$$

with

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \epsilon^{abc} A^b_M A^c_N \quad . \tag{4.88}$$

Note that we are allowed to choose and have chosen a specific normalization for the cosmological constant for simplicity, while the normalization of the gauge fields in the above corresponds to a definite value of coupling constant. In $n = 3$, this is dictated by the supersymmetry of $N = 3$ gauged supergravity [39], and we simply extend it to any dimensions. One can easily recover the gauge coupling constant dependence in our results below, if needed. The equations of motion are

$$\begin{aligned} & R_{MN} + \Big(n + \frac{1}{n-1} F^a_{PQ} F^{aPQ} \Big) g_{MN} - 2 F^a_{PM} F^{aP}{}_N = 0 \quad , \\ & \nabla_M F^{aM}{}_N + \epsilon^{abc} A^b_M F^{cM}{}_N = 0 \quad , \end{aligned} \tag{4.89}$$

where ∇_M is the covariant derivative with metric Christofel connections.

A charged black-brane solution in a general boosted frame is

$$\begin{aligned} ds^2 &= -r^2 V(r) u_\mu u_\nu dx^\mu dx^\nu - 2 u_\mu dx^\mu dr + r^2 \left(\eta_{\mu\nu} + u_\mu u_\nu \right) dx^\mu dx^\nu \quad , \\ A^a &= \sqrt{\frac{n-1}{2(n-2)}} \frac{q^a}{r^{n-2}} u_\mu dx^\mu \quad , \end{aligned} \tag{4.90}$$

[8] One can simply substitute ϵ^{abc} below with any structure constants of a Lie algebra to have the general results for arbitrary Lie groups.

with

$$V(r) = 1 - \frac{m}{r^n} + \frac{q^a q^a}{r^{2n-2}} \quad . \tag{4.91}$$

where it is simply obtained by embedding the $U(1)$ Reisner-Nordstrom black-brane into a Cartan direction inside $SU(2)$ which is specified by q^a ($a = 1, 2, 3$). As we are going to consider slow variations of q^a over the CFT spacetime x^μ, the Cartan $U(1)$ will correspondingly vary point-by-point, and the non-Abelian nature will manifest itself when these variations are not parallel to q^a locally.

Let us consider slowly varying parameters u_μ, m, and q^a up to first order, and we work in the frame where $u_\mu = (-1, 0, \cdots, 0)$ at the position $x^\mu = 0$. Then at first order in derivatives, we have

$$\begin{aligned} u_\mu &= (-1, x^\mu \partial_\mu u_i) \\ m &= m^{(0)} + x^\mu \partial_\mu m \\ q^a &= q^{a(0)} + x^\mu \partial_\mu q^a \quad , \end{aligned} \tag{4.92}$$

and the above black-brane solution will no longer be a solution with these varying parameters. To be a solution, we have to add corrections $g^{(1)}_{MN}$ and $A^{a(1)}_M$ to the zero'th order solution with varying parameters, which should be chosen to satisfy the equations of motion. Our gauge choice is as before;

$$g^{(1)}_{rr} = 0 \quad , \quad g^{(1)}_{r\mu} \sim u_\mu \quad , \quad A^{a(1)}_r = 0 \quad , \quad \sum_{i=1}^{n-1} g^{(1)}_{ii} = 0 \quad . \tag{4.93}$$

The resulting metric and the gauge fields at first order are

$$\begin{aligned} ds^2 &= -r^2 V^{(0)}(r) dt^2 + 2dtdr + r^2 (dx^i)^2 \\ &+ \left[x^\mu \left(\frac{(\partial_\mu m)}{r^{n-2}} - \frac{2q^a (\partial_\mu q^a)}{r^{2n-4}} \right) + g^{(1)}_{tt}(r) \right] dt^2 + 2 g^{(1)}_{tr}(r) dtdr \\ &+ 2 \left[x^\mu (\partial_\mu u_i) r^2 \left(V^{(0)}(r) - 1 \right) + g^{(1)}_{ti}(r) \right] dtdx^i + 2 \left[-x^\mu (\partial_\mu u_i) \right] drdx^i \\ &+ g^{(1)}_{ij}(r) dx^i dx^j \quad , \end{aligned} \tag{4.94}$$

$$\begin{aligned} A^a &= -\sqrt{\frac{n-1}{2(n-2)}} \frac{q^{a(0)}}{r^{n-2}} dt \\ &+ \left[-\sqrt{\frac{n-1}{2(n-2)}} \frac{1}{r^{n-2}} x^\mu (\partial_\mu q^a) + A^{a(1)}_t(r) \right] dt \\ &+ \left[\sqrt{\frac{n-1}{2(n-2)}} \frac{q^{a(0)}}{r^{n-2}} x^\mu (\partial_\mu u_i) + A^{a(1)}_i(r) \right] dx^i \quad . \end{aligned} \tag{4.95}$$

After a lengthy calculation, we get the following equations for the first order corrections $g_{MN}^{(1)}$ and $A_M^{a(1)}$. From the tt-part of Einstein equation,

$$
\begin{aligned}
&-\frac{1}{2}\frac{V(r)}{r^{n-3}}\partial_r\left(r^{n-1}\partial_r g_{tt}^{(1)}\right)-\frac{1}{2}r^2V(r)\partial_r\left(r^2V(r)\right)\left(\partial_r g_{tr}^{(1)}\right)-2n\left(r^2V(r)\right)g_{tr}^{(1)}\\
&-4(n-2)\sqrt{\frac{n-2}{2(n-1)}}\frac{q^a}{r^{n-3}}V(r)\left(\partial_r A_t^{a(1)}\right)\\
&=-\frac{1}{2}\partial_r\left(r^2V(r)\right)(\partial_i u_i)-\frac{(n-1)}{2r}\left(\frac{(\partial_t m)}{r^{n-2}}-\frac{2q^a\left(\partial_t q^a\right)}{r^{2n-4}}\right)\quad ,
\end{aligned}
\tag{4.96}
$$

from the tr-part,

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r g_{tt}^{(1)}\right)+\frac{1}{2}\partial_r\left(r^2V(r)\right)\left(\partial_r g_{tr}^{(1)}\right)+2ng_{tr}^{(1)}\\
&+4(n-2)\sqrt{\frac{n-2}{2(n-1)}}\frac{q^a}{r^{n-1}}\left(\partial_r A_t^{a(1)}\right)=\frac{(\partial_i u_i)}{r}\quad ,
\end{aligned}
\tag{4.97}
$$

from rr-part,

$$
\frac{(n-1)}{r}\left(\partial_r g_{tr}^{(1)}\right)=0\quad ,
\tag{4.98}
$$

and from $\sum_{i=1}^{n-1}(ii)$-part,

$$
\begin{aligned}
&\frac{(n-1)}{r^{n-3}}\partial_r\left(r^{n-2}g_{tt}^{(1)}\right)+(n-1)r^3V(r)\left(\partial_r g_{tr}^{(1)}\right)+2n(n-1)r^2g_{tr}^{(1)}\\
&-4\sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-3}}\left(\partial_r A_t^{a(1)}\right)=2(n-1)r\left(\partial_i u_i\right)\quad .
\end{aligned}
\tag{4.99}
$$

From the t-part of Maxwell equation, we have

$$
\begin{aligned}
\frac{V(r)}{r^{n-3}}\partial_r\left(r^{n-1}\partial_r A_t^{a(1)}\right) &- \sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-3}}V(r)\left(\partial_r g_{tr}^{(1)}\right)-\sqrt{\frac{n-1}{2(n-2)}}\epsilon^{abc}\frac{q^b}{r^{2n-4}}\partial_r\left(r^{n-2}A_t^{c(1)}\right)\\
&= -\sqrt{\frac{(n-1)(n-2)}{2}}\frac{1}{r^{n-1}}\left((\partial_t q^a)+q^a\left(\partial_i u_i\right)\right)\quad ,
\end{aligned}
\tag{4.100}
$$

and from the r-part,

$$
-\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r A_t^{a(1)}\right)+\sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-1}}\left(\partial_r g_{tr}^{(1)}\right)=0\quad .
\tag{4.101}
$$

The above six equations are scalar modes under $SO(n-1)$ spatial rotations.

Vector modes equations of $SO(n-1)$ are the following. From the ti-part of Einstein equation,

$$
\begin{aligned}
&-\frac{1}{2}\frac{V(r)}{r^{n-3}}\partial_r\left(r^{n+1}\partial_r\left(\frac{g_{ti}^{(1)}}{r^2}\right)\right)-2\sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-3}}V(r)\left(\partial_r A_i^{a(1)}\right)\\
&=\frac{1}{2}\frac{(\partial_i m)}{r^{n-1}}+\left(\frac{(n-1)}{2}rV(r)+\frac{n}{2}\frac{m}{r^{n-1}}\right)(\partial_t u_i)\quad ,
\end{aligned}
\tag{4.102}
$$

from the ri-part,

$$\frac{1}{2}\frac{1}{r^{n-1}}\partial_r\left(r^{n+1}\partial_r\left(\frac{g^{(1)}_{ti}}{r^2}\right)\right)+2\sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-1}}\left(\partial_r A_i^{a(1)}\right)=-\frac{(n-1)}{2r}\left(\partial_t u_i\right)\quad, \tag{4.103}$$

and from the i-components of Maxwell equation, we have

$$\begin{aligned}\frac{1}{r^{n-3}}\partial_r\left(r^{n-1}V(r)\partial_r A_i^{a(1)}\right) &+ \sqrt{\frac{(n-1)(n-2)}{2}}\frac{q^a}{r^{n-3}}\partial_r\left(\frac{g^{(1)}_{ti}}{r^2}\right)-2\sqrt{\frac{n-1}{2(n-2)}}\epsilon^{abc}\frac{q^b}{r^{n-\frac{5}{2}}}\partial_r\left(\frac{A_i^{c(1)}}{r^{\frac{1}{2}}}\right)\\ &= \sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-1}}\left(\left(\partial_i q^a\right)+q^a\left(\partial_t u_i\right)\right).\end{aligned} \tag{4.104}$$

Finally, the tensor mode, that is, traceless ij-components of Einstein equation is

$$-\frac{1}{2}\frac{1}{r^{n-3}}\partial_r\left(r^{n+1}V(r)\partial_r\left(\frac{g^{(1)}_{ij}}{r^2}\right)\right)=\frac{(n-1)}{2}r\left(\left(\partial_i u_j\right)+\left(\partial_j u_i\right)-\frac{2\delta_{ij}}{n-1}\left(\partial_k u_k\right)\right)\quad. \tag{4.105}$$

We now present the complete solution of the above equations.

4.1 The solution

We first solve the scalar mode equations. The constraint equations will be discussed after that. From (4.98), one finds that $g^{(1)}_{tr}=C$ with some constant C. To understand its meaning, note that this $g^{(1)}_{tr}=C$ will affect only (4.96), (4.97), (4.99), and it is easy to check that it can be compensated by turning on

$$g^{(1)}_{tt}=-2Cr^2\quad, \tag{4.106}$$

that is, C corresponds to the above homogeneous solution of the problem. As the above $g^{(1)}_{tt}$ is a non-normalizable perturbation to the boundary CFT metric (look at the r^2 factor in front), we see that C in fact corresponds to a non-normalizable homogeneous solution of the problem, and we set it zero. Then equation (4.101) is integrated to give us

$$A_t^{a(1)}=\frac{C^a}{r^{n-2}}+C^{a'}\quad, \tag{4.107}$$

where $C^{a'}=0$ is again a non-normalizable mode, and the meaning of C^a can be easily understood by looking back the zero'th order profile of the gauge fields

$$A_t^{a(0)}\sim\frac{q^a}{r^{n-2}}\quad, \tag{4.108}$$

that is, C^a is simply mapped to a redefinition of the charges q^a, so that we can also set it zero. Then one can easily integrate (4.99) to have

$$g_{tt}^{(1)} = \frac{2}{(n-1)} r\,(\partial_i u_i) + \frac{C'}{r^{n-2}} \quad , \tag{4.109}$$

with an integration constant C'. However, recall that the zero'th order $g_{tt}^{(0)}$ is

$$g_{tt}^{(0)} = -r^2 V(r) = -r^2 + \frac{m}{r^{n-2}} - \frac{q^a q^a}{r^{2n-4}} \quad , \tag{4.110}$$

so that C' is simply a redefinition of the energy density m. In summary, the only non-vanishing scalar mode in the solution is

$$g_{tt}^{(1)} = \frac{2}{(n-1)} r\,(\partial_i u_i) \quad . \tag{4.111}$$

One then finds the following three kinds of constraints equations,

$$\begin{aligned}
0 &= g^{rt}E_{tt} + g^{rr}E_{rt} = \frac{(n-1)}{2r^{n-1}}\left((\partial_t m) + \frac{n}{(n-1)} m\,(\partial_i u_i)\right) - \frac{(n-1)q^a}{r^{2n-3}}\left((\partial_t q^a) + q^a\,(\partial_i u_i)\right) \quad , \\
0 &= g^{rt}M_t^a + g^{rr}M_r^a = \sqrt{\frac{(n-1)(n-2)}{2}}\frac{1}{r^{n-1}}\left((\partial_t q^a) + q^a\,(\partial_i u_i)\right) \quad , \\
0 &= g^{rt}E_{ti} + g^{rr}E_{ri} = -\frac{1}{2r^{n-1}}\left((\partial_i m) + nm\,(\partial_t u_i)\right) \quad ,
\end{aligned} \tag{4.112}$$

which results in

$$(\partial_t m) = -\frac{n}{(n-1)} m\,(\partial_i u_i) \quad , \quad (\partial_i m) = -nm\,(\partial_t u_i) \quad , \quad (\partial_t q^a) = -q^a\,(\partial_i u_i) \, . \tag{4.113}$$

As before, they are the conservation laws for the zero'th order energy-momentum tensor and the $SU(2)$ currents;[9]

$$\begin{aligned}
T^{\mu\nu(0)} &= \frac{m}{16\pi G_{n+1}}\left(\eta^{\mu\nu} + n u^\mu u^\nu\right) \equiv p\left(\eta^{\mu\nu} + n u^\mu u^\nu\right) \quad , \\
J^{\mu a(0)} &= \frac{1}{4\pi G_{n+1}}\sqrt{\frac{(n-1)(n-2)}{2}} q^a u^\mu \equiv \rho^a u^\mu \quad .
\end{aligned} \tag{4.114}$$

We next solve for the vector modes, Eqs.(4.103) and (4.104), which will turn out to have an important new ingredient due to the non-Abelian nature. An inspection shows that the only chance to see non-Abelian nature at this order is when $(\partial_i q^a)$ is not parallel to q^a. Define

$$Q_i^a \equiv \epsilon^{abc} q^b\,(\partial_i q^c) \quad , \tag{4.115}$$

[9]We will obtain them more rigorously in the next section.

then the following three $SU(2)$ vectors in the Lie algebra of $SU(2)$ form a normal basis;

$$\left\{q^a, Q_i^a, \epsilon^{abc} q^b Q_i^c\right\} \quad , \tag{4.116}$$

and especially one can expand $(\partial_i q^a)$ in terms of them as

$$(\partial_i q^a) = \left(\frac{\vec{q}\cdot(\partial_i \vec{q})}{\vec{q}\cdot\vec{q}}\right) q^a + \left(-\frac{1}{\vec{q}\cdot\vec{q}}\right)\epsilon^{abc} q^b Q_i^c \quad , \tag{4.117}$$

where $\vec{P}\cdot\vec{Q} \equiv P^a Q^a$. We then have to expand our $A_i^{a(1)}$ in terms of them as

$$A_i^{a(1)} = A_i^{(1)}(r) q^a + f^{(1)}(r) Q_i^a + g^{(1)}(r)\epsilon^{abc} q^b Q_i^c \quad , \tag{4.118}$$

with radial functions $A_i^{(1)}$, $f^{(1)}$, and $g^{(1)}$ to be determined. Observe that we have dropped the index i for $f^{(1)}$ and $g^{(1)}$ as we will see that they are independent of i[10]. Plugging this expansion into our equations (4.103) and (4.104), one obtains the following two equations for $A_i^{(1)}$ and $g_{ti}^{(1)}$,

$$\frac{1}{2}\frac{1}{r^{n-1}}\partial_r\left(r^{n+1}\partial_r\left(\frac{g_{ti}^{(1)}}{r^2}\right)\right) + 2\sqrt{\frac{(n-1)(n-2)}{2}}\frac{\vec{q}\cdot\vec{q}}{r^{n-1}}\left(\partial_r A_i^{(1)}\right) = -\frac{(n-1)}{2r}\left(\partial_t u_i\right), \tag{4.119}$$

$$\begin{aligned}\frac{1}{r^{n-3}}\partial_r\left(r^{n-1}V(r)\partial_r A_i^{(1)}\right) + \sqrt{\frac{(n-1)(n-2)}{2}}\frac{1}{r^{n-3}}\partial_r\left(\frac{g_{ti}^{(1)}}{r^2}\right) \\ = \sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-1}}\left(\frac{\vec{q}\cdot(\partial_i\vec{q})}{\vec{q}\cdot\vec{q}} + (\partial_t u_i)\right) \quad ,\end{aligned} \tag{4.120}$$

and the following coupled equations for $f^{(1)}$ and $g^{(1)}$,

$$\frac{1}{r^{n-3}}\partial_r\left(r^{n-1}V(r)\partial_r f^{(1)}\right) + 2\sqrt{\frac{n-1}{2(n-2)}}\frac{\vec{q}\cdot\vec{q}}{r^{n-\frac{5}{2}}}\partial_r\left(\frac{g^{(1)}}{r^{\frac{1}{2}}}\right) = 0\,, \tag{4.121}$$

$$\frac{1}{r^{n-3}}\partial_r\left(r^{n-1}V(r)\partial_r g^{(1)}\right) - 2\sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-\frac{5}{2}}}\partial_r\left(\frac{f^{(1)}}{r^{\frac{1}{2}}}\right) = -\sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-1}}\frac{1}{\vec{q}\cdot\vec{q}}\,. \tag{4.122}$$

It is straightforward to solve (4.119) and (4.120) as is done in the STU model, and we simply present the result;

$$\begin{aligned} g_{ti}^{(1)} &= r^2 V(r)\int_\infty^r dr'\,\frac{1}{(r')^{n+1}(V(r'))^2}\left(\int_{r_H}^{r'} dr''\,I(r'') - r_H^{n-1}V'(r_H)C_i\right) \quad , \qquad (4.123) \\ A_i^{(1)} &= \int_\infty^r dr'\,\frac{1}{(r')^{n-1}V(r')}\left(Q_i(r') - Q_i(r_H) - \sqrt{\frac{(n-1)(n-2)}{2}}\left(\frac{g_{ti}^{(1)}(r')}{(r')^2} - \frac{C_i}{r_H^2}\right)\right) \quad , \end{aligned}$$

[10]This should be clear from the spatial $SO(3)$ symmetry because Q_i^a and $\epsilon^{abc}q^b Q_i^c$ are already $SO(3)$ vectors.

where

$$\begin{aligned}
Q_i(r) &= -\sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r}\left(\frac{\vec{q}\cdot(\partial_i\vec{q})}{\vec{q}\cdot\vec{q}}+(\partial_t u_i)\right) \quad , \\
I(r) &= -4\sqrt{\frac{(n-1)(n-2)}{2}}\frac{\vec{q}\cdot\vec{q}}{r^{n-1}}\left(Q_i(r)-Q_i(r_H)+\sqrt{\frac{(n-1)(n-2)}{2}}\frac{C_i}{r_H^2}\right) \\
&- (n-1)r^{n-2}V(r)\,(\partial_t u_i)\ .
\end{aligned}$$

Again, the integration constant C_i is fixed by the Landau frame constraint, and an explicit computation gives us

$$C_i = \frac{r_H^2}{nm}\Bigg(-\frac{2}{(n-2)r_H^{n-1}}\,(\vec{q}\cdot\partial_i\vec{q})+\left(nr_H^{n-1}+\frac{(n^2-4n+2)\vec{q}\cdot\vec{q}}{(n-2)r_H^{n-1}}\right)(\partial_t u_i)\Bigg).$$

However, we are unable to integrate the equations (4.121) and (4.122) to solve $f^{(1)}$ and $g^{(1)}$; we will instead comment on a possible numerical approach. Defining $F^{(+)(1)} \equiv f^{(1)}+i|\vec{q}|g^{(1)}$, the equations (4.121) and (4.122) become a single complex equation

$$\frac{1}{r^{n-3}}\partial_r\left(r^{n-1}V(r)\partial_r F^{(+)(1)}\right)-2i|\vec{q}|\sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-\frac{5}{2}}}\partial_r\left(\frac{F^{(+)(1)}}{r^{\frac{1}{2}}}\right)=-i|\vec{q}|\sqrt{\frac{n-1}{2(n-2)}}\frac{1}{r^{n-1}}\frac{1}{\vec{q}\cdot\vec{q}}.$$

The equation has one trivial solution,

$$F^{(+)(1)}=-\frac{1}{\vec{q}\cdot\vec{q}} \quad , \tag{4.124}$$

which is non-normalizable. However, having this solution is of great help in finding the unique normalizable solution regular at the horizon numerically; one only solves the *homogeneous* equation without the source term in the right-hand side, and then add the above trivial solution to have a normalizable solution. Considering the limit of the *homogeneous* equation to the horizon where $V(r_H)=0$, one gets the relation

$$\left.\frac{\left(\partial_r F^{(+)(1)}\right)}{F^{(+)(1)}}\right|_{r=r_H}=\frac{-i|\vec{q}|\sqrt{n-1}}{\sqrt{2(n-2)}r_H^{n+1}V'(r_H)-2i\sqrt{n-1}|\vec{q}|r_H} \quad , \tag{4.125}$$

for a regular homogeneous solution. Putting $F^{(+)(1)}(r_H)=1$, the above completely specifies the boundary condition at the horizon, and one can numerically solve the differential equation uniquely, that is, regular homogeneous solution normalized as $F^{(+)(1)}(r_H)=1$ is unique. Let's call this homogeneous solution $F_0^{(+)(1)}(r)$. In general, its large r asymptotic will give us a finite non-zero constant $F_0^{(+)(1)}(\infty)\neq 0$, and one constructs the full solution of our original problem simply as

$$F^{(+)(1)}(r)=-\frac{1}{\vec{q}\cdot\vec{q}}\left(1-\frac{F_0^{(+)(1)}(r)}{F_0^{(+)(1)}(\infty)}\right) \quad . \tag{4.126}$$

Finally, the tensor mode equation (4.105) is easily integrated to give us

$$g_{ij}^{(1)} = -2\sigma_{ij} r^2 \int_\infty^r dr' \, \frac{1}{(r')^{n+1} V(r')} \left((r')^{n-1} - (r_H)^{n-1} \right) \quad , \tag{4.127}$$

with

$$\sigma_{ij} = \frac{1}{2} \left((\partial_i u_j) + (\partial_j u_i) - \frac{2\delta_{ij}}{n-1} (\partial_k u_k) \right) \quad . \tag{4.128}$$

4.2 The first order transport coefficients

Based on the results of the previous section, we can write the covariant form of the metric and the gauge field up to first order in derivative expansion. The metric looks as

$$\begin{aligned} ds^2 &= -r^2 V(r) u_\mu u_\nu dx^\mu dx^\nu - 2u_\mu dx^\mu dr + r^2 P_{\mu\nu} dx^\mu dx^\nu + \frac{2}{n-1} r(\partial_\rho u_\rho) u_\mu u_\nu dx^\mu dx^\nu \\ &- r^2 V(r) u_\mu \Big(\int_\infty^r dr' \frac{1}{(r')^{n-1} V^2(r')} \left(\int_{r_H}^{r'} dr'' I_\nu(r'') - r_H^{n-1} V'(r_H) C_\mu \right) \Big) dx^\mu dx^\nu \\ &- 2r^2 \sigma_{\mu\nu} \Big(\int_\infty^r dr' \frac{1}{(r')^{n+1} V(r')} \Big((r')^{n-1} - (r_H)^{n-1} \Big) \Big) dx^\mu dx^\nu \quad , \end{aligned} \tag{4.129}$$

where the first-order functions $I_\mu^{(1)}(r)$ and $Q_\mu^{(1)}(r)$ are defined respectively as

$$\begin{aligned} I_\mu^{(1)}(r) &= -4\sqrt{\frac{(n-1)(n-2)}{2}} \frac{q.q}{r^{n-1}} \left(Q_\mu^{(1)}(r) - Q_\mu^{(1)}(r_H) + \sqrt{\frac{(n-1)(n-2)}{2}} \frac{C_\mu}{r_H^2} \right) \\ &+ (n-1) r^{n-2} V(r) u^\nu \partial_\nu u_\mu \quad , \\ Q_\mu^{(1)}(r) &= -\sqrt{\frac{n-1}{2(n-2)}} \frac{1}{r} \Big(\frac{q \cdot P_\mu^\nu \partial_\nu q}{q \cdot q} - u^\nu \partial_\nu u_\mu \Big) \quad , \end{aligned} \tag{4.130}$$

and we also defined

$$\sigma^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \frac{P^{\mu\nu}}{n-1} (\partial_\alpha u^\alpha) \quad . \tag{4.131}$$

The gauge field is found as

$$\begin{aligned} A^a &= q^a \Big(\sqrt{\frac{n-1}{2(n-2)}} \frac{1}{r^{n-2}} u_\mu + \int_\infty^r dr' \frac{1}{(r')^{n-1} V(r')} \Big(Q_\mu^{(1)}(r') - Q_\mu^{(1)}(r_H) \\ &+ \sqrt{\frac{(n-1)(n-2)}{2}} \frac{C_\mu}{r_H^2} \Big) - \sqrt{\frac{(n-1)(n-2)}{2}} \int_\infty^r dr' \frac{1}{(r')^{n-1}} \\ &\quad \int_\infty^{r'} dr'' \frac{1}{(r'')^{n+1} V^2(r'')} \left(\int_{r_H}^{r''} dr''' I_\mu^{(1)}(r''') - r_H^{n-1} V'(r_H) C_\mu \right) \Big) dx^\mu \\ &+ \left(f^{(1)}(r) Q_\mu^a + g^{(1)}(r) \epsilon^{abc} q^b Q_\mu^c \right) dx^\mu \quad , \end{aligned}$$

where $Q^a_\mu \equiv \epsilon^{abc} q^b P^\nu_\mu (\partial_\nu q^c)$. Note that the last line is the non-Abelian induced terms.

Again using holographic renormalization, one can easily find the stress tensor and the $SU(2)$ charge current. In terms of Fefferman-Graham coordinate expansion,

$$g_{\mu\nu}(\rho) = \eta_{\mu\nu} + \cdots + \frac{g^{(n)}_{\mu\nu}}{\rho^n} + \cdots \quad , \tag{4.132}$$

the n-dimensional CFT stress tensor is given by

$$T_{\mu\nu} = \frac{n}{16\pi G_{n+1}} g^{(n)}_{\mu\nu} \quad . \tag{4.133}$$

Again, it is straightforward to check that the only non-vanishing first order correction comes from $g^{(1)}_{ij}$ with

$$T^{(1)}_{ij} = -2\eta\sigma_{ij} \quad , \tag{4.134}$$

where

$$\eta = \frac{r_H^{n-1}}{16\pi G_{n+1}} = \frac{s}{4\pi} \quad . \tag{4.135}$$

We point out that it is not a trivial fact here and in the STU model that there is no first order correction to T^{00}, as it results from a non-trivial cancellations in the expansion. The full covariant expression of the stress tensor is

$$T^{\mu\nu} = p\left(\eta^{\mu\nu} + n u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu} + \cdots \quad . \tag{4.136}$$

with the pressure $p = \frac{m}{16\pi G_{n+1}}$ representing the zero'th order contribution.

The $SU(2)$ charge current is similarly obtained from the expansion as

$$J^a_\mu = \frac{(n-2)}{4\pi G_{n+1}} \lim_{\rho\to\infty} \rho^{n-2} A^a_\mu(\rho) \quad , \tag{4.137}$$

and it is easy to find the first order correction to be

$$J^{a(1)}_\mu = \frac{1}{4\pi G_{n+1}} \left(Q^{(1)}_\mu(r_H) - \sqrt{\frac{(n-1)(n-2)}{2}} \frac{C_\mu}{r_H^2} \right) q^a + \frac{(n-2)}{4\pi G_{n+1}} \left(f^{(n-2)} Q^a_\mu + g^{(n-2)} \epsilon^{abc} q^b Q^c_\mu \right),$$

where $Q^{(1)}_\mu(r)$ is given before in (4.130), and $f^{(n-2)}$, $g^{(n-2)}$ are the coefficients of $1/\rho^{n-2}$ in the expansion of $f^{(1)}(\rho)$ and $g^{(1)}(\rho)$ respectively. Trading q^a with the density ρ^a defined in (4.114), we can rewrite the result in the form

$$\begin{aligned} J^{a(1)}_\mu &= -\mathcal{D}\left(\frac{\rho\cdot P^\nu_\mu(\partial_\nu \rho)}{\rho\cdot\rho} - u^\nu \partial_\nu u_\mu \right)\rho^a + \mathcal{D}_1 \epsilon^{abc} \rho^b P^\nu_\mu (\partial_\nu \rho^c) \\ &+ \mathcal{D}_2 P^\nu_\mu \left(\rho^a(\rho\cdot\partial_\nu\rho) - (\rho\cdot\rho)(\partial_\nu \rho^a)\right) \quad , \end{aligned} \tag{4.138}$$

with three diffusion coefficients

$$
\begin{aligned}
\mathcal{D} &= \frac{1}{(n-2)r_H}\left(1-\frac{2\vec{q}\cdot\vec{q}}{nmr_H^{n-2}}\right)=\frac{(n-2)m+2r_H^n}{n(n-2)mr_H} \quad , \\
\mathcal{D}_1 &= \frac{8\pi G_{n+1}f^{(n-2)}}{(n-1)} \quad , \quad \mathcal{D}_2=\frac{2^{\frac{11}{2}}\pi^2 G_{n+1}^2 g^{(n-2)}}{(n-1)^{\frac{3}{2}}(n-2)^{\frac{1}{2}}} \quad .
\end{aligned} \tag{4.139}
$$

The $\mathcal{D}$ is essentially the usual diffusion coefficient of Abelian nature, which agrees with ref.[40], while the other two diffusion coefficients are due to the non-Abelian properties. Although their precise values can only be determined by numerical analysis, we hope that the above structure of non-Abelian current we obtain in derivative expansion may be important for future applications to the QCD plasma incorporating non-Abelian symmetry.

4.3 On Tri-Sasakian compactification of M-theory to AdS_4

We would like to conclude by a few comments on the realization of our $SU(2)$ theory in a concrete example. Our 4-dimensional bosonic action ($n = 3$) with $SU(2)$ gauge symmetry in the bulk has a specific AdS_4/CFT_3 realization in M-theory; consider N M2 branes sitting at the apex of an 8-dimensional Hyper-Kahler cone and take a near horizon limit. The superconformal theory one gets on the M2 branes has $N = 3$ or 6-real components of dynamical supersymmetry with $SU(2)_R$ R-symmetry. The corresponding dual theory on AdS_4 will include a consistent truncation to the minimal $N = 3$ $SU(2)$ gauged supergravity in 4-dimensions[39], whose bosonic action is precisely our action in the previous section. In this case, one has an explicit expression for the Newton's constant G_4 in terms of the number of M2 branes as follows.

The 11-dimensional M-theory supergravity action is

$$
\mathcal{L}_{11}=\frac{1}{(2\pi)^8 l_p^9}\int d^{11}x\,\sqrt{-g_{11}}\left(R^{(11)}-\frac{1}{2}|F_4|^2\right)-\frac{1}{6(2\pi)^8 l_p^9}\int C_3\wedge F_4\wedge F_4 \quad , \tag{4.140}
$$

where $R^{(11)}$ is the Ricci scalar of the 11-dimensional metric, and

$$
|F_4|^2=\frac{1}{4!}F_{MNPQ}F^{MNPQ} \quad . \tag{4.141}
$$

The near horizon limit of M2 branes at the tip of a Hyper-Kahler cone takes a form of $AdS_4 \times X_7$ with X_7 being a Tri-Sasakian 7-fold which is the unit radius section of the Hyper-Kahler cone involved. The explicit solution is given as

$$
\begin{aligned}
ds_{11}^2 &= R^2\left(\frac{1}{4}ds_{AdS_4}^2+d\Omega_{X_7}^2\right) \\
F_4 &= \frac{3}{8}R^3\epsilon_4 \quad ,
\end{aligned} \tag{4.142}
$$

where $d\Omega^2_{X_7}$ is the metric of X_7 normalized in such a way that

$$R_{ab} = 6g_{ab} \quad , \tag{4.143}$$

and ϵ_4 is the volume form of the unit radius AdS_4. The constant R is given by the relation

$$6R^6 \mathrm{vol}(X_7) = (2\pi l_p)^6 N \quad . \tag{4.144}$$

Then one can easily obtain the 4-dimensional effective action on AdS_4 after compactifying M-theory action (4.140) on the above 7-dimensional Tri-Sasakian manifold X_7. Note that the X_7 metric of $R^2 d\Omega^2_{X_7}$ now has

$$R^{X_7}_{ab} = \frac{6}{R^2} g^{X_7}_{ab} \quad , \tag{4.145}$$

so that $R^{X_7} = \frac{42}{R^2}$, which means

$$R^{(11)} \sim R^{(4)} + \frac{42}{R^2} \quad . \tag{4.146}$$

Also one has

$$|F_4|^2 = \left(\frac{3}{8}R^3\right)^2 \left(\frac{4}{R^2}\right)^4 = \frac{36}{R^2} \quad . \tag{4.147}$$

Combined with

$$\int d^{11}x \sqrt{-g_{11}} = R^7 \mathrm{vol}(X_7) \int d^4x \sqrt{-g_4} \quad , \tag{4.148}$$

the effective 4-dimensional action takes a form

$$\frac{R^7 \mathrm{vol}(X_7)}{(2\pi)^8 l_p^9} \int d^4x \sqrt{-g_4} \left(R^{(4)} + \frac{24}{R^2} + \cdots\right) \quad , \tag{4.149}$$

and we identify

$$\frac{1}{16\pi G_4} = \frac{R^7 \mathrm{vol}(X_7)}{(2\pi)^8 l_p^9} \quad . \tag{4.150}$$

We then need to put $R = 2$ to conform to our convention of cosmological constant in the previous section[11], and using (4.144) one finally has

$$\frac{1}{16\pi G_4} = \frac{2\pi N^{\frac{3}{2}}}{2^2 6^{\frac{3}{2}} \left(\mathrm{vol}(X_7)\right)^{\frac{1}{2}}} \quad . \tag{4.151}$$

For a class of Tri-Sasakian manifolds that are obtained from Hyper-Kahler quotients, their normalized volumes $\mathrm{vol}(X_7)$ are explicitly known[41, 42]. One can start from a

[11] The quicker way to arrive at this conclusion is to make the AdS_4 in (4.142) to have unit radius, which should be a solution with our cosmological constant convention.

$(2+r)$-dimensional flat quaternion space and take $U(1)^r$ Hyper-Kahler quotients specified by charges Q_a^i where i runs over $U(1)$ and a runs over $(2+r)$ quaternions. The resulting 8-dimensional Hyper-Kahler cone will have a unit radius section as a Tri-Sasakian manifold, whose normalized volume is known by the formula[42]

$$\mathrm{vol}\,(X_7) = \frac{2^{r+1}\pi^4}{\Gamma(4)\mathrm{Vol}\,(U(1)^r)} \int \prod_{i=1}^{r} d\phi^i \prod_{a=1}^{2+r} \frac{1}{1+\left(\sum_{i=1}^{r} Q_a^i \phi^i\right)^2} \quad . \tag{4.152}$$

In the simplest example of $r=1$ with three charges Q_i $(i=1,2,3)$, it becomes

$$\mathrm{vol}\,(X_7(Q_1,Q_2,Q_3)) = \frac{\pi^4}{3}\frac{(Q_1Q_2+Q_2Q_3+Q_3Q_1)}{(Q_1+Q_2)(Q_2+Q_3)(Q_3+Q_1)} \quad , \tag{4.153}$$

which includes the famous $N(1,1)$ as $Q_1=Q_2=Q_3=1$ with

$$\mathrm{vol}\,(N(1,1)) = \frac{\pi^4}{8} \quad . \tag{4.154}$$

Although the $N=3$ superconformal theory on M2 branes dual to the gravity background with $N(1,1)$ is still unknown, there is a proposal in ref.[43] for the case of $Q_1 \pm Q_2 \pm Q_3 = 0$, inspired by the theory of BL/ABJM[44, 45]. See also ref.[46] for $N=2$ cases.

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A Appendix 1

A convenient expression for the first order variation of the Ricci tensor we use is

$$\delta R_{MN} = -\nabla_M C^P_{PN} + \nabla_P C^P_{MN} \quad , \tag{A.155}$$

with

$$C^P_{MN} = \frac{1}{2} g^{PQ} \left(\nabla_M \delta g_{NQ} + \nabla_N \delta g_{MQ} - \nabla_Q \delta g_{MN}\right) \quad , \tag{A.156}$$

where ∇_M is the covariant derivative with respect to the zero'th order metric g_{MN}. The tensor C^P_{MN} is in fact a variation of the metric Christoffel symbol, $C^P_{MN} = \delta\Gamma^P_{MN}$.

B Appendix 2

We denote

$$A(r) = H^{-\frac{2}{3}}(r)f(r) \quad , \quad B(r) = H^{-\frac{1}{6}}(r) \quad , \quad C(r) = r^2 H^{\frac{1}{3}}(r) \quad , \tag{B.157}$$

as before, where the local parameters m and q_I are implicit in the expressions above and below. The prime in the equations below means the radial derivative $\frac{d}{dr}$.

$$P_i^{(1)} = \frac{1}{4r^3 H(r)} \left(H(r) \left(\sum_{I=1}^{3} \frac{\partial_i q_I}{H_I^2(r)} \right) - \left(\sum_{I=1}^{3} H_I(r) \right) \left(\sum_{J=1}^{3} \partial_i q_J \right) + \sum_{I=1}^{3} \left(H_I(r) \partial_i q_I \right) \right) ,$$

$$S_{tt}^{(1)} = -\frac{3A'}{4BC}(\partial_t C) - A\left(\frac{1}{B^2}(\partial_t B) \right)' - \left(\frac{B'}{B} + \frac{3C'}{2C} \right) \frac{A}{B^2}(\partial_t B) + \frac{3C'}{4BC}(\partial_t A) - \frac{A'}{2B}(\partial_i u_i) ,$$

$$S_{tr}^{(1)} = \left(\frac{1}{B}(\partial_t B) + \frac{3}{2C}(\partial_t C) \right)' + \frac{3C'}{4C^2}(\partial_t C) + \frac{1}{2} \sum_{I=1}^{3} \frac{1}{(X^I)^2} \left(\partial_r X^I \right) \left(\partial_t X^I \right) + \frac{C'}{2C}(\partial_i u_i) ,$$

$$\sum_i S_{ii}^{(1)} = \frac{3}{B}(\partial_t C)' + \frac{3C'}{2BC}(\partial_t C) + \frac{3C'}{B}(\partial_i u_i) ,$$

$$\begin{aligned} S^{1(1)} &= -\frac{1}{B}\left(\frac{(\partial_t X^1)}{X^1} + \frac{1}{2}\frac{(\partial_t X^2)}{X^2} \right)' - \left(\frac{1}{B}\left(\frac{(\partial_t X^1)}{X^1} + \frac{1}{2}\frac{(\partial_t X^2)}{X^2} \right) \right)' \\ &- \frac{1}{B}\left(\frac{B'}{B} + \frac{3C'}{2C} \right)\left(\frac{(\partial_t X^1)}{X^1} + \frac{1}{2}\frac{(\partial_t X^2)}{X^2} \right) - \frac{3}{2BC}(\partial_t C)\left(\frac{(X^1)'}{X^1} + \frac{1}{2}\frac{(X^2)'}{X^2} \right) \\ &- \frac{1}{B}\left(\frac{(X^1)'}{X^1} + \frac{1}{2}\frac{(X^2)'}{X^2} \right)(\partial_i u_i) . \end{aligned}$$

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